Constraint solving with a learning mechanism based on general constraints

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Research Thesis

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy

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Submitted to the Senate
of the Technion — Israel Institute of Technology
Tamuz 5774 Haifa June 2014

The research thesis was done under the supervision of Assoc. Prof. Ofer Strichman

List of Publications

- Chapter 2 is a reprint of the published paper [VS10a]: *A Proof-Producing CSP Solver.*

  Author contributions: CSP conflict analysis, lazy constraint explanations, proofs of unsatisfiability, various deduction rules used in proofs.

- The technical report [VS10b] extending [VS10a] is in Appendix B: *A Proof-Producing CSP Solver (A proof supplement).*

  Author contributions: Proofs of correctness for [VS10a], node relaxation[1] of conflict clauses, augmented constraint explanations.

- Chapter 3 is a technical report [VS14] of a paper under submission: *Learning non-clausal constraints in CSP (long version).*

  Author contributions: Non-clausal conflict analysis, the Combine rule used in conflict analysis.

- The source code of the HCSP CSP solver:
  `http://tx.technion.ac.il/~mveksler/HCSP/` .

  Author contributions: independently written a CSP solver that, beyond implementing all of the published algorithms, can generate a Craig interpolant.

The generous financial support of the Technion is gratefully acknowledged.

[1] In the original paper the term *rejuvenation* was used instead of relaxation

Contents

List of Figures

List of Tables

List of Algorithms

Abstract

The Constraint Satisfaction Problem (CSP) is a known NP-hard problem in the field of Artificial Intelligence. As part of this thesis, I developed a new CSP solver called HCSP, that has several novel capabilities and a reasoning engine that is highly competitive and on many benchmarks the fastest available. It is the first CSP solver capable of generating machine-checkable proofs in case the formula is unsatisfiable (based on an inference system called signed resolution), and also the first CSP solver that is able to generate a Craig interpolant. Both of these capabilities were so far only available in SAT solvers, and the main challenge in adapting them to CSP is the need to bridge between the general constraints comprising the input problem, and an inference system that provides the building blocks of a proof.

HCSP uses novel algorithms that work directly on CSP (i.e., not via translation to CNF SAT like some other CSP solvers) and are able to learn stronger constraints than a SAT solver. During conflict analysis, it is able to learn new non-clausal constraints. For example, from $x \leq y$ and $y \leq z$ it may infer that $x \leq z$, although $x \leq z$ is not an existing predicate in the formula (in contrast to theory propagation in SMT solvers). This capability is based on a new general inference rule, Combine, which can be seen as a generalization of CNF resolution.

The thesis is comprised of a collection of new algorithms and of known algorithms adapted from other domains (mostly SAT) to the realm of CSP.

Abbreviations and Notations

Notation	**Page(s)**
CSP	5
$\mathcal{C}$	5
$\mathcal{D}$	5
$\mathcal{V}$	5
$\perp$	34,66

Chapter 1

Introduction

1.1 The Constraint Satisfaction Problem (CSP)

Many problems in engineering can be formulated as making choices that have to satisfy a set of constraints. When scheduling flight crews, one needs to consider the constraint that a passenger aircraft may fly only, e.g., with a pilot and 3 assistants on board and that they are not scheduled at the same time on another flight. When choosing a route for a car there are constraints on the possible routes and speed limit. Many times there are complicated interactions between constraints. Consider for example the problem of scheduling lectures in a university. Lectures schedule is constrained by personal preferences of lecturers, rooms that cannot accommodate more than one lecture simultaneously, students that cannot attend more than one lecture at a time, curriculum requirements, and many more. Other real-life examples of problems involving constraints are automated test generation for hardware circuits, nurse shift scheduling, container placement on a ship, car configuration, software-application dependency and update analysis, and many more.

The problem of finding a solution that satisfies all the constraints is called the Constraint Satisfaction Problem (CSP). CSP is a known NP-hard problem (we discuss complexity in some detail in Sect. 1.1.1), and is difficult in practice for a computer because the input of real problems is typically large.

Formally, a Constraint Satisfaction Problem (CSP) is defined by a triple $\langle \mathcal{C}, \mathcal{V}, \mathcal{D} \rangle$ where $\mathcal{V}$ is a set of variables, $\mathcal{D}$ is their domains, i.e., for each variable a set of its allowed values, and $\mathcal{C}$ is a set of constraints over these variables. Effectively, a domain $D_v \in \mathcal{D}$ can be viewed as the unary constraint $v \in D_v$. A feasible solution is an assignment to the variables that satisfies all the constraints in C.

One may observe that special cases of CSP are SAT and Pseudo-Boolean (PB) problems, both of which are still NP-complete, but require a relatively simpler algorithm to

solve. In SAT all the variables are restricted to be Boolean. In PB the variables are Boolean but their values are interpreted as 0 and 1 rather than false and true, respectively, and linear constraints over these variables are allowed, e.g., $2x_1 + 3x_2 \leq 2$, where x_1, x_2 are Boolean. There are many use cases for both SAT and PB in the industry, and there are many dedicated solvers for these special cases of CSP.

In this work, as most other works on CSP, domains are restricted to be finite sets of integral values. For example, consider a graph coloring problem where we have to color the nodes of Figure 1.1 in two colors such that no adjacent edges share the same color. In this example we have four nodes numbered 1..4 and edges between them $\{(1,2),(1,3),(2,4),(2,3),(3,4)\}$. The CSP is modeled by four variables representing the colors of the four nodes, where 1 stands for red and 2 for green. The problem is then:

$$v_1 \neq v_2 \wedge v_1 \neq v_3 \wedge v_2 \neq v_4 \wedge v_3 \neq v_4 \quad .$$

This can be solved, e.g., with $v_1 \leftarrow 1,\ v_2 \leftarrow 2,\ v_3 \leftarrow 2, v_4 \leftarrow 1$.

A previously popular representation of CSP was a graph, where variables are represented as nodes and constraints as arcs. This view is consistent with older works such as [Mac77] where a constraint may affect only two variables. The graph representation of the example graph-coloring CSP is shown in Figure 1.1.

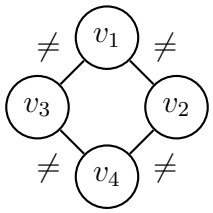


Figure 1.1: Example with a graph-coloring CSP

This limitation to binary constraints is unlike newer works, including this one, where a constraint may affect any number of variables. Such a constraint is sometimes called a *global constraint* or *generalized constraint*, and a CSP with global constraints is sometimes called a *Generalized CSP*. A graph model of a generalized CSP uses hyper-arcs to represent the constraints, i.e., arcs that have more than two ends.

1.1.1 A few words about complexity

In [FV98] there is a proof that a CSP with n-ary constraints is NP-complete. This proof has a hidden assumption, which may not hold, that the time complexity for checking a constraint is polynomial. Although this is usually the case, there is nothing in our

definition of CSP[1] that guarantees this. In theory, it is possible to formulate a constraint whose checking complexity is NP-complete, or worse.

For example, consider the following #P-complete global constraint, which cannot be decomposed into a polynomial number of simpler constraints. Given a set of variables $v_1, \ldots, v_n$ and a constant u, the constraint is satisfied if the assignment to $v_1, \ldots, v_n$ corresponds to a CNF that has not more than u solutions. Clearly propagating this constraint is in #P.

We now explain how an assignment to $v_1, \ldots, v_n$ can be interpreted as a particular CNF. These variables represent the stream of literals of the formula reminiscent of the DIMACS format. An assignment such as $v_4 = 2$ indicates that the fourth literal of the CNF is x_2. Similarly an assignment such as $v_6 = -3$ indicates that the sixth literal is $\neg x_3$. Finally, 0 indicates the end of the clause, e.g., $v_5 = 0$ indicates an end of a clause at the fifth location.

Let us demonstrate this encoding with a particular example. Let $v_1, \ldots, v_7$ be the set of variables in the constraint, and $u = 5$ be the bound. Then the following assignment:

$$v_1 = 2, v_2 = -3, v_3 = 0, v_4 = -2, v_5 = 3, v_6 = 1, v_7 = 0$$

is a solution, because it corresponds to the CNF

$$(x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3 \vee x_1) \ ,$$

which has 5 satisfying assignments.

On one hand, answering the question whether the constraint is satisfied is in #P, not in NP. On the other hand, there are too many valid and invalid assignments for pre-computing an explicit relation. This means that a problem with this constraint is not in NP and it is impossible to formulate it as an NP problem. This implies that although CSP can represent NP-complete problems, its complexity may be outside of NP. In fact, it has no upper bound on complexity. Despite this, to the best of my knowledge, all constraints supported by CSP solvers in the public domain can be checked in polynomial time leading to NP complexity for solving practical CSP.

The following subsections explain general concepts of CSP and CSP solving through their relation to this research and their implementation in our solver HCSP.

[1]Our definition of constraints allows them to be formulated by any means, including by a Turing machine. However, when constraints are defined as (pre-computed) relations then it is possible to implement constraint-checking in $O(n)$ where n is the number of columns in the relation.

1.2 The CSP solver

The HaifaCSP constraint solver, or HCSP for short, was written during this research in order to explore new solving techniques. It is able to solve complex CSPs with many different types of constraints. HCSP was written afresh in C++ with no reliance on external source code (the only exception being a toolkit for parsing the XCSP-1.2 input format [RL09] used for parsing CSP benchmarks used in [DLR09]). HCSP is comprised of 60k lines of code, total, and only 25k if tests, comments, empty lines, braces and semicolons are excluded.

HCSP supports four input formats:

- XCSP-2.1 input format [RL09] used in the 2009 CSP competition [DLR09].
- MiniZinc [MSKS10] format without sets or floating point arithmetic.
- A subset of the OPB input format [MR+06] used in pseudo-Boolean optimization.
- A C-like input format with most of the C-language's expression language, without side-effects. This input format has also support for minimization/maximization of an objective function and setting of variable domains.

HCSP supports most of the satisfiability problems of the 2009 CSP competition [DLR09]. The only exception are problems with *element* and *cumulative* constraints which are described in [BCR05]. The optimization directive of the XCSP-2 format, unlike with MiniZinc format, is not supported. Appendix A has a list of constraints which HCSP supports.

1.3 The CSP solving algorithm

The core of HCSP's algorithm is based on MAC-3 [Mac77] (Maintain Arc Consistency) which is commonly used in CSP solvers. Although the original MAC-3 was defined over binary constraints, i.e., constraining pairs of variables, it was extended to n-ary constraints. Although many publications refer to this extension as *Generalized-MAC-3* (GMAC-3)[2], we will simply call it MAC from this point on.

A slightly different MAC algorithm was described in [SE97]. A non-recursive variant of this algorithm is described in Algorithm 1.1 and serves as the basis for the HCSP solving algorithm. MAC detects a conflict on line 4, which leads simple backtracking without conflict analysis. The simple backtrack-search may lead to repetitions of the

[2]Some publications refer to *Generalized-MAC* as GAC and others as GMAC

same bad decisions (*thrashing*). This intuition has been reinforced by [Mit03] showing how good conflict-analysis speeds up the solver.

Algorithm 1.1 The non-recursive MAC solving algorithm

```
 1: function MAC(C, V, D)
 2:     while true do
 3:         D ←PROPAGATE (C, V, D);          ▷ If ANY-EMPTY (D) then do nothing.
 4:         if ANY-EMPTY (D) then
 5:             backtrack_level ← current_level − 1;
 6:             if backtrack_level ≥ 0 then
7.1:                 (var, value) ←decision_at_level(current_level);
7.2:                 D ←BACKTRACK-TO (backtrack_level);
7.3:                 D_var ← D_var \ {value};          ▷ This may empty the domain.
 8:             else
 9:                 return UNSAT;
10:             end if
11:         else if ALL-ASSIGNED (D) then
12:             return The solution is in D;
13:         else
14:             D ←DECIDE (D);
15:         end if
16:     end while
17: end function
```

Unlike MAC, EFC [BK] and HCSP analyze conflicts and learn new constraints directly on the CSP. The solving algorithms of HCSP is depicted in Algorithm 1.2 and in Figure 1.2.

The difference between MAC in Algorithm 1.1 and HCSP in Algorithm 1.2 lies in the way conflicts are handled. At line 4 a conflict is detected, in both algorithms. At line 5 both algorithms calculate the backtracking level, i.e., the decision level to backtrack to[3]. MAC simply backtracks one level, while HCSP generates a conflict constraint and uses the constraint when deciding on the next backtrack level in line 5.

The HCSP backtracking process is simpler than the non-recursive MAC. In Algorithm 1.1 the additional lines 7.1, 7.3 are required to avoid repeating the same conflict, lines which are unnecessary in Algorithm 1.2. In Algorithm 1.2 the same conflict is avoided by learning a conflict constraint, instead of explicitly negating the decision.

[3]A decision level l is a state at which there are l decisions. When backtracking to level l all domain modifications done since, and including, decision $l+1$, are reverted.

Algorithm 1.2 The HCSP solving algorithm

```
1: function SOLVE-CSP(C, V, D)
2:     while true do
3:         D ←PROPAGATE (C, V, D);
4:         if ANY-EMPTY (D) then
5:             backtrack_level ← CSP-ANALYZE-CONFLICT (C, V, D);
6:             if backtrack_level ≥ 0 then
7:                 D ←BACKTRACK-TO (backtrack_level);
8:             else
9:                 return UNSAT;
10:            end if
11:        else if ALL-ASSIGNED (D) then
12:            return The solution is in D;
13:        else
14:            D ←DECIDE (D);
15:        end if
16:    end while
17: end function
```

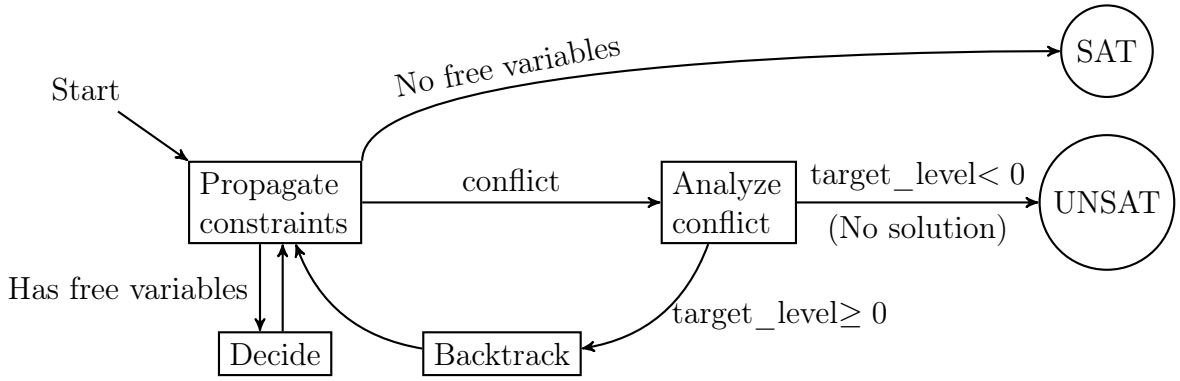


Figure 1.2: The solving loop in HCSP

1.3.1 Propagating constraints

Constraint Propagation is the part of MAC that examines constraints and reduces variable domains according to the constraints. In SAT this phase is called BCP (Boolean Constraint Propagation). During this phase the solver examines constraints, one at a time, and removes values from the domains that cannot satisfy the constraint.

Consider, for example, the CSP

$$D_{x_1} = \{2, 3, 4\},\ D_{x_2} = \{1, 2, 3\}, \quad x_1 \leq x_2$$

In this case neither $x_1 = 4$ nor $x_2 = 1$ can participate in a satisfying assignment. We say that neither $x_1 = 4$ nor $x_2 = 1$ are supported by $x_1 \leq x_2$ in D_{x_1} and D_{x_2}. When the solver

propagates[4] $x_1 \leq x_2$ it removes the unsupported values from the domains, producing new domains:

$$D'_{x_1} = \{2, 3\},\ D'_{x_2} = \{2, 3\} \quad .$$

With D'_{x_1} and D'_{x_2} the constraint $x_1 \leq x_2$ becomes *arc-consistent*[5]. A constraint is arc-consistent if it supports all values in the current domains. Formally, constraint $c(x_1, \ldots, x_n)$ is arc-consistent if:

$$\forall i \in \{1, 2, \ldots, n\} \forall x''_i \in D_{x_i} \left[\exists x'_1 \in D_{x_1} \exists x'_2 \in D_{x_2} \cdots \exists x'_n \in D_{x_1}. \left[x'_i = x''_i \wedge c(x'_1, \ldots, x'_n)\right]\right] \ .$$

The CSP is said to be *arc-consistent* if all its constraints are arc-consistent. When the CSP is arc-consistent then it is no longer possible to deduce domain reductions by examining one constraint at a time.

The algorithm that propagates a given constraint is the *propagator* for that constraint. If a propagator always removes all values which are unsupported by the constraint then it is called *precise*. An imprecise propagator performs an over-approximated propagation, potentially leaving some unsupported values in the domains. An imprecise propagator never under-approximates, i.e., it never removes a possible solution to the problem.

A propagator may be imprecise for several reasons:

- Algorithmic complexity of precise propagation may be too high to be practical. For example, the constraint catalog [BCR05] contains `two_layer_edge_crossing`, `global_cardinality`, and `weighted_partial_alldiff` whose precise propagation algorithms are NP-hard.

- Memory complexity of precise propagation may be too high. For example, for domain $D_{x_1} = D_{x_2} = D_{x_3} = \{2, 3, \ldots, 10^9\}$ and constraint $x_1 = x_2 * x_3$ precise propagation should reduce the domain of x_1 to

  $$D'_{x_1} = \{x' \in \mathbb{N} | 2 \leq x' \leq 10^9 \wedge \neg \text{IS\_PRIME}(x')\} \quad .$$

  It is every difficult to define a set representation that can hold this domain efficiently.

- Overall efficiency may be better if the propagator is approximated. For example, the imprecise propagation given by *bounds consistency* [Dec03] has a simple and fast implementation since it has to deal only with domain bounds.

MAC propagates constraints until no propagator reduces any domain. If all propagators are precise then the propagation stops when the CSP is arc-consistent. Even though

[4] In some publications, such as [Mac77], the term *Revise* is used instead of *Propagate* to denote the propagation step that makes a single constraint arc-consistent.

[5] The term *arc* comes from the graph representation of CSP, which was mentioned in Section 1.1.

the solver will possibly not reach arc-consistency with imprecise propagators, it is still guaranteed to stop. The guarantee comes from the domains being finite and allowing a finite number of reductions, and since the propagation algorithm stops when nothing is propagated.

1.3.2 Decision Making

After MAC reaches arc-consistency, the solver selects a variable according to some variable-ordering heuristics and assigns it a value from its domain according to some value-ordering heuristics. This assignment operation is called a *decision*. Note that it is possible that this decision leads to a dead-end, i.e., there is no solution to the CSP constrained by the decision.

In the following example the CSP is arc-consistent:

$$D_{x_1} = \{1, 2, 3\}, \quad D_{x_2} = D_{x_3} = \{1, 2\}, \quad x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge x_2 \neq x_3 \quad .$$

Assume that the solver decides $x_1 \leftarrow 1$, which makes $x_1 \neq x_2$ reduce the domain to $D'_{x_2} = \{2\}$ and $x_1 \neq x_3$ reduce the domain to $D'_{x_2} = \{2\}$. Then $x_2 \neq x_3$ detects a conflict, i.e., no solution is possible with these domains. The solver backtracks, undoing the decision, concludes that $x_1 = 1$ cannot participate in any solution, and removes it from the domain. This produces a reduced CSP:

$$D_{x_1} = \{2, 3\}, \quad D_{x_2} = D_{x_3} = \{1, 2\}, \quad x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge x_2 \neq x_3 \quad .$$

With this newly reduced problem, if the solver decides $x \leftarrow 2$ then it also reaches a conflict. While undoing the decision it concludes that $D_{x_1} = \{3\}$. Next it decides $x_2 \leftarrow 1$ which leads to the solution $x_1 = 3,\ x_2 = 1,\ x_3 = 2$.

Notice that this example has two very similarly bad decisions with similar conflicts. It would be good if the solver could detect the similarity and avoid taking the second bad decision. In order for this to happen the solver has to perform conflict analysis, which is discussed in Subsection 1.3.3.

1.3.3 Conflict Analysis

The CSP solution process involves making wrong decisions and backtracking from them. Many CSP solving algorithms were proposed that avoid repeating similar wrong decisions. Initial methods used a no-good to avoid a bad assignment, which works well in solvers that do not have constraint propagation such as BT-search [Dec03]. There were several failed attempts to have conflict analysis in solvers that propagate constraints such as MAC-CBJ [Pro95], which looks for the decision combination that causes a conflict.

It has been shown in [BR96] that MAC-CBJ is not competitive with plain MAC. In order to overcome these issues works such as [JDB00] learned no-goods to avoid repeating bad decisions efficiently. For example, a nogood $(x_1 \leftarrow 1, x_2 \leftarrow 5, x_4 \leftarrow 8)$ means that an assignment with $x_1 = 1 \wedge x_2 = 5 \wedge x_4 = 8$ is impossible and has to be avoided.

Nogoods did not catch on and the reason, as explained in [KB05, Mit03], was that no-goods are too weak to efficiently represent causes for a conflict. Instead of restricting conflict analysis to detecting bad assignments, [KB05] proposes to extend nogoods into *generalized-nogoods*, or *g-nogoods* for short. A g-nogood adds an *anti-assignment* operator $x \not\leftarrow n$ to indicate that there would be a conflict only if $x \neq n$. For example, a g-nogood $(x_1 \not\leftarrow 1, x_1 \not\leftarrow 2, x_2 \leftarrow 5)$ means that an assignment with $x_1 \neq 1 \wedge x_1 \neq 2 \wedge x_2 = 5$ is impossible and has to be avoided. This can be expressed as constraint $(x_1 \in \{1, 2\} \vee x_2 \neq 5)$. The EFC [BK] solver would explain every reduced domain in terms of g-nogoods, eagerly, and resolve these g-nogoods during conflict analysis to get the cause of the conflict in terms of a g-nogood.

Another competing technique is called C-Res, i.e., *Constraint-Resolution*, (introduced in [Mit03]). Each time a constraint is propagated and values are removed, a Boolean clause[6] is created to represent this propagation. When performing conflict analysis, the clauses representing the constraints are resolved using the standard propositional resolution. It is unspecified whether the clauses are generated during constraint propagation or only during conflict analysis.

A later technique named *Lazy* [7] *Clause Generation* [OSC09], combines CSP solving techniques with SAT. This technique represents all domains and CSP variables as Boolean indicator variables and perform solving on them. It examines a constraint and if it is inconsistent with the domains then the solver generates an explanation of the inconsistency, i.e., a Boolean-CNF clause which is added to the CNF. The propagation is then performed via BCP [MMZ+01b] and so is conflict analysis and learning are performed on the SAT side. The effects of this approach should be similar to the eager generation of g-nogoods in EFC with the difference that SAT solvers manage propagation and learning more efficiently.

Both lazy clause generation [OSC09] and EFC [BK] produce explanations eagerly. This creates a big collection of clauses (or g-nogoods) which may be never required for conflict analysis or for the solution. Another possible disadvantage of [OSC09] is its relatively inefficient domain representation. Each value of variable's domain requires a Boolean variable, which makes it impractical to represent big domains. In order to be

[6] In later works, such as [OSC09], this Boolean clause is named an explanation clause

[7] *Lazy* refers to the fact that the CNF encoding for the whole CSP is not created up-front. The CNF encoding is created one bit at a time, i.e, lazily, when the constraints are examined in the context of the current partial assignment.

able to maintain big domains it is possible to represent them using a $\log_2$ representation, which requires auxiliary variables. This idea was introduced, to an extent, in [AS12].

HCSP extends on these notions and overcomes the inherent limitations of *lazy clause generation* and EFC's g-nogoods. It uses *many valued SAT* [BHM00b] to efficiently represent big domains and to learn clauses that affect them. Unlike the competing techniques it generates explanations during conflict analysis and not during propagation. This lets HCSP take less memory and to learn not only clauses and g-nogoods but also complicated constraints. The description of the HCSP conflict analysis is given in the next chapter.

1.4 Research contribution

The thesis is structured as a collection of (extended-versions of) an article [VS10a] published in AAAI'10, an article currently under submission [VS14], and a proof supplement that was published as a technical report [VS10b]. Chapter 4 includes additional material that is not included in these publications. The main contributions of this research are summarized in the following list:

- **A fast CSP solver**. HCSP is a new solver written as part of this research. It is the fastest[8] solver running the benchmarks of the 2009 CSP competition [DLR09], the last one held using the original format of this solver. Transition to the CSP input language of **MiniZinc** also shows promising results when comparing them to other solvers supporting the **MiniZinc** language.

- **Lazy explanation generation**. In Chap. 2 we explain how HCSP refines the basic concept of EFC [BK, KB05] and *Lazy Clause Generation* [OSC09], i.e., explaining a propagation via a clause, and performs it only during conflict analysis. The lazy generation has a lower memory cost. Lazy clause generation makes it possible to choose the best possible explanation for a given propagation, when there is more than one possibility. Moreover, it allows to generate an *augmented explanation*, as explained in Subsection B.4.2. An augmented explanations does not explain the propagation, but rather explains why propagating a constraint leads to a conflict.

- **Combine**. Algorithm 3.1 in Subsection 3.2.2 introduces the notion of *constraint combination* which generalizes binary resolution of many-valued SAT (which is a generalization of binary resolution of Boolean SAT). Introduced in Subsection 3.4.1

[8] HCSP is the fastest if benchmarks with table constraints are excluded. The current implementation of table constraints is inefficient and has to be rewritten.

the *combine* rule for c_1 and c_2 and pivot x is defined as:

$$\frac{c_1(x,\bar{y}) \qquad c_2(x,\bar{y})}{x \notin \mathcal{X} \vee \exists x' \in \mathcal{X}.[c_1(x',\bar{y}) \wedge c_2(x',\bar{y})]} \quad (\text{ combine(x) }) \quad .$$

Where $\bar{y}$ is a collection of variables which are constrained by either c_1, c_2, or both and $D_{\bar{y}}$ is their domain. Also:

$$\mathcal{X} = \{x' \in \mathcal{Z} | \forall \bar{y'} \in D_{\bar{y}}.[\neg c_1(x',\bar{y'}) \vee \neg c_2(x',\bar{y'})]\} \quad .$$

The research shows that *combine* possess properties which make it work well in conflict analysis. Also, several strong rules are derived using the general combine rule.

- **Non-clausal conflict analysis.** Using the *combine* rule HCSP employs a novel conflict-analysis algorithm which learns complicated constraints. It is able, for example, to combine $x_1+2x_2+7x_3 \geq 0$ with $x_1+3x_2 \geq 5$ and get $2x_1+5x_2+7x_3 \geq 5$, and then to combine the result with $2x_1 + 5x_2 + 7x_3 \leq 4$ and detect that these constraints contradict with no need for further backtracking. Other conflict analysis methods such as [BK, KB05, VS10a] require conflicts of explanation clauses to reach the same conclusion.

  By having specialized combination rules only for some pairs of constraints, HCSP slightly resembles SMT.[9] The SMT solver groups constraints into separate *theories*, processed independently. Much of the inference between constraints is performed independently in each theory, similarly to the combination rules that work only on specific pairs of constraints. But unlike HCSP, SMT inference is mostly visible by asserting existing constraints through conflict analysis and theory propagation, and not by introducing new constraints during conflict analysis.

  Other major differences between SMT and HCSP are the decision procedure and propagation. In SMT decisions are made on Boolean variables that encode more complicated constraints: should an atom, i.e., constraint, be satisfied or falsified in order to satisfy the overall formula. In HCSP the decisions are on all variables. In SMT, theory propagation may learn a new Boolean constraint that affects existing constraints (atoms). In HCSP, propagation reduces the domains of variables and only conflict analysis may introduce new constraints, including Boolean constraints that affect exiting constraints.

[9] SMT and CSP traditionally do not focus on the same set of problems, although there are some problems that are dealt with by both. Whereas CSP is typically focused on finite discrete domains, SMT can accommodate any decidable theory, including those over variables of type Real.

- **Proofs**. Section 2.4 describes how HCSP generates machine-checkable proofs for unsatisfiability using basic inference rules. It uses *explanation* rules to generate signed-clauses out of constraints, and a binary signed-resolution to create new clauses. HCSP is the only CSP solver known to us that generates proofs, a feature that was so far only available in SAT solvers.

- **Interpolants**. HCSP can generate interpolants, and is the only CSP solver known to us that has this capability. In Section 4.5 we prove the correctness of Algorithm 4.4, which finds Craig's Interpolant for a CSP. The algorithm is based on a similar algorithm that is used by some SAT solvers for the case of propositional formulas and binary resolution.

Chapter 2

A Proof-Producing CSP Solver[1]

Abstract

HCSP *is a CSP solver that can produce a machine-checkable deductive proof in case it decides that the input problem is unsatisfiable. The roots of the proof may be nonclausal constraints, whereas the rest of the proof is based on resolution of signed clauses, ending with the empty clause.* HCSP *uses parameterized, constraint-specific inference rules in order to bridge between the nonclausal and the clausal parts of the proof. The consequent of each such rule is a signed clause that is 1) logically implied by the nonclausal premise, and 2) strong enough to be the premise of the consecutive proof steps. The resolution process itself is integrated in the learning mechanism, and can be seen as a generalization to CSP of a similar solution that is adopted by competitive SAT solvers.*

2.1 Introduction

Many problems in planning, scheduling, automatic test-generation, configuration and more, can be naturally modeled as Constraint Satisfaction Problems (CSP) [Dec03], and solved with one of the many publicly available CSP solvers. The common definition of this problem refers to a set of variables over finite and discrete domains, and arbitrary constraints over these variables. The goal is to decide whether there is an assignment to the variables from their respective domains, which satisfies all the constraints. If the answer is positive the assignment that is emitted by the CSP solver can be verified easily. On the other hand a negative answer is harder to verify, since current CSP solvers do not produce a deductive proof of unsatisfiability.

[1]Published in AAAI-10

In contrast, most modern CNF-based SAT solvers accompany an unsatisfiability result with a deductive proof that can be checked automatically. Specifically, they produce a *resolution proof*, which is a sequence of application of a single inference rule, namely the binary *resolution rule*. In the case of SAT the proof has uses other than just the ability to independently validate an unsatisfiability result. For example, there is a successful SAT-based model-checking algorithm which is based on deriving interpolants from the resolution proof [HJMM04].

Unlike SAT solvers, CSP solvers do not have the luxury of handling clausal constraints. They need to handle constraints such as $a < b + 5$, $allDifferent(x, y, z)$, $a \neq b$, and so on. However, we argue that the effect of a constraint in a given state can always be replicated with a *signed clause*, which can then be part of a resolution proof. A signed clause is a disjunction between *signed literals*. A signed literal is a unary constraint, constraining a variable to a domain of values. For example, the signed clause $(x_1 \in \{1, 2\} \vee x_2 \notin \{3\})$ constrains[2] x_1 to be in the range $[1, 2]$ or x_2 to be anything but 3. A conjunction of signed clauses is called *signed CNF*, and the problem of solving signed CNF is called *signed SAT*[3], a problem which attracted extensive theoretical research and development of tools [LKM03, BHM00b].

In this article we describe how our arc-consistency-based CSP solver HCSP (for a "Proof-producing Constraint Solver") produces deductive proofs when the formula is unsatisfiable. In order to account for propagations by general constraints it uses constraint-specific parametric inference rules. Each such rule has a constraint as a premise and a signed clause as a consequent. These consequents, which are generated during conflict analysis, are called *explanation clauses*. These clauses are logically implied by the premise, but are also strong enough to imply the same literal that the premise implies at the current state. The emitted proof is a sequence of inferences of such clauses and application of special resolution rules that are tailored for signed clauses.

Like in the case of SAT, the signed clauses that are learned as a result of analyzing conflicts serve as 'milestone' atoms in the proof, although they are not the only ones. They are generated by a repeated application of the resolution rule. The intermediate clauses that are generated in this process are discarded and hence have no effect on the solving process itself. In case the learned clause eventually participates in the proof HCSP reconstructs them, by using information that it saves during the learning process. We will describe this conflict-analysis mechanism in detail in Section 2.3 and 2.4, and compare it to alternatives such as 1-UIP [ZMMM01], MVS [LKM03] and EFC [KB05]

[2]Alternative notations such as $\{1, 2\}{:}x_1$ and $x_1^{\{1,2\}}$ are used in the literature to denote a signed literal $x_1 \in \{1, 2\}$.

[3]Signed SAT is also called MV-SAT (i.e. Many Valued SAT).

in Section 2.5. We begin, however, by describing several preliminaries such as CSP and signed SAT, and by introducing our running example.

2.2 Preliminaries

2.2.1 The Constraint Satisfaction Problem (CSP)

A CSP is a triplet $\phi = \langle \mathcal{V}, \mathcal{D}, \mathcal{C} \rangle$, where $\mathcal{V} = \langle v_1, \ldots, v_n \rangle$ is the set of problem variables, $\mathcal{D} = \langle D_1, \ldots, D_n \rangle$ is the set of their respective domains and $\mathcal{C}$ is the set of constraints over these variables. An assignment α satisfies a CSP ϕ if it satisfies all the constraints in $\mathcal{C}$ and $\forall v_i \in \mathcal{V}.\alpha(v_i) \in D_i$. A CSP is unsatisfiable if there is no assignment that satisfies it.

We will use the example below as our running example.

Example 2.2.1 *Consider three intervals of length 4 starting at a, b and c. The CSP requires that these intervals do not overlap and fit a section of length 11. This is clearly unsatisfiable as the sum of their lengths is 12. The domains of a,b and c is defined to be* $[1, 8]$. *It is clear that in this case the domains do not impose an additional constraint, since none of these variables can be assigned a value larger than 8 without violating the upper-bound of 11.*

In addition our problem contains three Boolean variables x_1, x_2, x_3 *that are constrained by* $x_1 \vee x_2,\ x_1 \vee x_3,\ (x_2 \wedge x_3) \rightarrow a = 1$. *Although the problem is unsatisfiable even without the constraints over these variables, we add them since the related constraints will be helpful later on for demonstrating the learning process and showing a proof of unsatisfiability that refers only to a subset of the constraints.*

We use $NoOverlap(a, L_a, b, L_b)$ *to denote the constraint* $a + L_a \leq b \vee b + L_b \leq a$. *Overall, then, the formal definition of the CSP is:*

$$\mathcal{V} = \{a, b, c, x_1, x_2, x_3\};$$

$$\mathcal{D} = \begin{cases} D_a = D_b = D_c = [1, 8], \\ D_{x_1} = D_{x_2} = D_{x_3} = \{0, 1\}; \end{cases}$$

$$\mathcal{C} = \begin{cases} c_1 : NoOverlap(a, 4, b, 4) & c_4 : x_1 \vee x_2 \\ c_2 : NoOverlap(a, 4, c, 4) & c_5 : x_1 \vee x_3 \\ c_3 : NoOverlap(b, 4, c, 4) & c_6 : (x_2 \wedge x_3) \rightarrow a = 1\ . \end{cases}$$

2.2.2 Signed clauses

A *signed literal* is a unary constraint, which means that it is a restriction on the domain of a single variable. A positive signed literal, e.g., $a \in \{1, 2\}$, indicates an allowed range

of values, whereas a negative one, e.g., $a \notin [1,2]$ indicates a forbidden range of values. When the domain of values referred to by a literal is a singleton, we use the equality and disequality signs instead, e.g., $b = 3$ stands for $b \in \{3\}$ and $b \neq 3$ stands for $b \notin \{3\}$. A *signed clause* is a disjunction of signed literals. For brevity we will occasionally write *literal* instead of *signed literal* and *clause* instead of *signed clause*.

Propagation of signed clauses

Signed clauses are learned at run-time and may also appear in the original formulation of the problem. Our solver HCSP has a propagator for signed clauses, which is naturally based on the *unit clause rule*. A clause with n literals such that $n-1$ of them are false and one is unresolved is called a unit clause. The unit clause rule simply says that the one literal which is unresolved must be asserted. After asserting this literal other clauses may become unit, which means that a chain of propagations can occur.

A clause is ignored if it contains at least one satisfied literal. If all the literals in a clause are false then propagation stops and the process of *conflict analysis* and *learning* begins. We will consider this mechanism in Section 2.3.

Resolution rules for signed clauses

HCSP uses Beckert et al.'s generalization of binary resolution to signed clauses in order to generate resolution-style proofs [BHM00a]. They defined the *signed binary resolution* and *simplification* rules, and implicitly relied on a third rule which we call *join literals*. In the exposition of these rules below, X and Y consist of a disjunction of zero or more literals, whereas A and B are sets of values.

signed binary resolution

$$\frac{(v \in A \vee X) \quad (v \in B \vee Y)}{(v \in (A \cap B) \vee X \vee Y)}[R_v]$$

simplification

$$\frac{(v \in \emptyset \vee Z)}{(Z)}[S_v]$$

join literals

$$\frac{((\bigvee_i v \in A_i) \vee Z)}{(v \in (\bigcup_i A_i) \vee Z)}[J_v]$$

Resolution over signed clauses gives different results for a different selection of the pivot variable. For example:

$$\frac{(a \in \{1,2\} \vee b \in \{1,2\}) \qquad (a = 3 \vee b = 3)}{(a \in \emptyset \vee b \in \{1,2\} \vee b = 3)}[R_a]$$

$$\frac{(a \in \emptyset \vee b \in \{1,2\} \vee b = 3)}{(b \in \{1,2\} \vee b = 3)}[S_a] \qquad \frac{(b \in \{1,2\} \vee b = 3)}{(b \in \{1,2,3\})}[J_a]$$

gives a different result if the pivot is b instead of a:

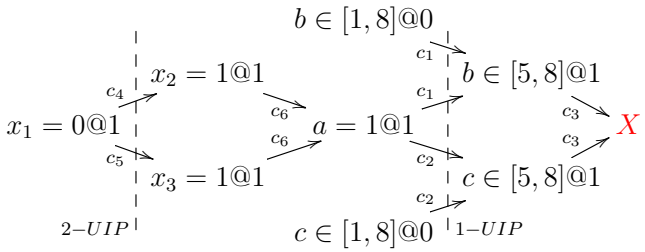


Figure 2.1: An implication graph corresponding to the running example.

$$\frac{(a \in \{1,2\} \vee b \in \{1,2\}) \qquad (a = 3 \vee b = 3)}{(a \in \{1,2,3\})}[R_b + S_b + J_b] \ .$$

In practice HCSP does not list applications of the 'join-literals' and 'simplification' rules, simply because they are applied very frequently and it is possible to check the proof without them, assuming this knowledge is built into the proof checker. Such a checker should apply these rules until convergence after each resolution, in order to create the premise of the next step.

2.3 Learning

In this section we explain the learning mechanism in HCSP, and how it is used for deriving proofs of unsatisfiability based, among other things, on the resolution rules that were defined in the previous section. We begin with implication graphs, which are standard representation of the propagation process. In Section 2.3.2 we will show the conflict analysis algorithm.

2.3.1 Implication graphs and conflict clauses

A propagation process is commonly described with an *implication graph*. Figure 2.1 shows such a graph for our running example, beginning from the decision $x_1 = 0$. In this graph vertices describe domain updates. For example the vertex labeled with $b \in [5,8]@1$ means that the domain of b was updated to $[5,8]$ at decision level 1. A special case is a vertex labeled with an initial domain, which may only occur at decision level 0, e.g., $b \in [1,8]@0$. A conflict between clauses is signified by X. Directed edges show logical implications, and are annotated with the implying constraint. The incoming edges of each node are always labeled with the same constraint.

A constraint is called *conflicting* if it is evaluated to *false* by the current assignment. In our example the constraint $c_3 = NoOverlap(b, 4, c, 4)$ is conflicting under the assignment

$x_1 = 0$. When such a constraint is detected the solver has to analyze the conflict and infer its cause. Traditionally this process has two roles: to apply learning by producing a new constraint and to select a decision level that the solver should backtrack to. The learned constraint has to be logically implied by the formula, and to forbid, as a minimum, the assignment that caused the conflict. In practice the goal is to produce a more general constraint, such that a larger portion of the search space is pruned. In competitive SAT solvers and CSP solvers such as EFC, the constraint is built such that it necessarily leads to further propagation right after backtracking (this constraint is called an *asserting clause* in SAT).

A standard technique for performing conflict analysis in SAT, which can also be used in CSP is called 1-UIP (for 'first Unique Implication Point') [ZMMM01]. The dashed line marked as 1-UIP in Figure 2.1 marks a cut in the graph that separates the conflict node from the decision and assignments in previous decision levels. There is only one vertex immediately to the left of the line — namely the node labeled with $a = 1@1$ — which is both on the current decision level and has edges crossing this line to the right-hand side. Nodes with this property are called UIPs. UIPs, in graph-theory terms, are *dominators* of the conflicting node with respect to the decision. In other words, all paths from the decision to the conflicting node must go through each UIP. A UIP is called 1-UIP if it is the rightmost UIP. 2-UIP marks the second UIP from the right, etc.

Asserting all the literals immediately on the left of a cut necessarily leads to a conflict. For example, collecting the literals on the left of the 1-UIP cut in Figure 2.1 shows that $(a = 1 \wedge b \in [1,8] \wedge c \in [1,8])$ imply a conflict. To avoid the conflict the solver can generate a *conflict clause* that forbids this combination, namely $(a \neq 1 \vee b \notin [1,8] \ \vee c \notin [1,8])$ in this case. HCSP produces stronger clauses than those that can be inferred by 1-UIP, by using resolution combined with a simplification step. This is the subject of the next subsection.

2.3.2 Conflict analysis and learning

Algorithm 2.1 describes the conflict analysis function in HCSP, which is inspired by the corresponding function in SAT [ZM03]. This algorithm traverses the implication graph from right to left, following backwards the propagation order.

We will use the following notation in the description of the algorithm. For a node u, let $lit(u)$ denote the literal associated with u. For a literal l, let $var(l)$ denote the variable corresponding to l. For a set of nodes U, let $vars(U) = \{var(lit(u)) \mid u \in U\}$, and for a clause[4] cl, let $vars(cl) = \{var(l) \mid l \in cl\}$.

[4]Here we use the standard convention by which a clause can also be seen as a set of literals.

	Constraint	Explanation clause
c_1	NoOverlap(a,4,b,4)	$\omega_1 = (a \notin [1,4] \vee b \notin [1,4])$
c_2	NoOverlap(a,4,c,4)	$\omega_2 = (a \notin [1,4] \vee c \notin [1,4])$
c_3	NoOverlap(b,4,c,4)	$\omega_3 = (b \notin [5,8] \vee c \notin [5,8])$
c_4	$x_1 \vee x_2$	$\omega_4 = (x_1 \neq 0 \vee x_2 \neq 0)$
c_5	$x_1 \vee x_3$	$\omega_5 = (x_1 \neq 0 \vee x_3 \neq 0)$
c_6	$(x_2 \wedge x_3) \rightarrow a = 1$	$\omega_6 = (x_2 \neq 1 \vee x_3 \neq 1 \vee a = 1)$

Table 2.1: Constraints and explanation clauses for the running example. The explanation clauses refer to the inferences depicted in the implication graph in Figure 2.1.

A key notion in the algorithm is that of an *explanation clause*:

Definition 2.3.1 (Explanation clause) *Let u be a node in the implication graph such that $lit(u) = l$. Let $(l_1, l) \ldots (l_n, l)$ be the incoming edges of u, all of which are labeled with a constraint r. A signed clause c is an* explanation clause *of a node u if it satisfies:*

1. $r \rightarrow c$,
2. $(l_1 \wedge \cdots \wedge l_n \wedge c) \rightarrow l$.

We can see from the definition that an explanation clause is strong enough to make the same propagation of the target literal given the same input literals. Note that if the constraint r happens to be a clause, then the notions of explanation clause and *antecedent clause* that is used in SAT, coincide.

Example 2.3.1 *Explanation clauses for our running example appear in the third column in Table 2.1. These clauses are built with respect to the nodes in the implication graph in Figure 2.1. We will explain how they are generated in Section 2.3.3.*

The algorithm begins by computing *cl*, an explanation clause for the conflicting node *conflict-node*. In line 3 it computes the predecessor nodes of *conflict-node* and stores them in *pred*. The function RELEVANT ($\langle nodes \rangle$, $\langle clause \rangle$) that is invoked in line 4 returns a subset N of the nodes in $\langle nodes \rangle$ that are relevant for the clause $\langle clause \rangle$, i.e., $vars(N) = vars(\langle clause \rangle)$.

Let *dl* be the latest decision level in *front*. The loop beginning in line 5 is guarded by STOP-CRITERION-MET (*front*), which is true in one of the following two cases:

- There is a single node in level *dl* in *front*, or
- $dl = 0$, and none of the nodes in *front* has an incoming edge.

At each iteration of the loop, CSP-Analyze-Conflict updates *cl* which, in the end of this process, will become the conflict clause. The set *front* maintains the following invariant just before line 6: *The clause cl is inconsistent with the labels in* front. Specifically, in each iteration of the loop, CSP-Analyze-Conflict :

- assigns the latest node in *front* in the propagation order to *curr-node*, and removes it from *front*,
- finds an explanation *expl* clause to *curr-node*,
- resolve the previous clause *cl* with *expl*, where $var(lit(\textit{curr-node}))$ is the resolution variable (the resolution process in line 9 is as was explained in Section 2.2.2), and
- adds the predecessors of *curr-node* to *front* and removes redundant nodes as explained below.

The input to the function Distinct is a set of nodes $\langle nodes \rangle$. It outputs a maximal subset N of those such that no two nodes are labeled with the same variable. More specifically, for each variable $v \in vars(\langle nodes \rangle)$, let $U(v)$ be the maximal subset of nodes in $\langle nodes \rangle$ that are labeled with v, i.e., for each $u \in U(v)$ it holds that $var(lit(u)) = v$. Then N contains only the right-most node on the implication graph that is in $U(v)$.

The invariance above and other properties of Algorithm 2.1 are proved in [VS10b].

Algorithm 2.1 Conflict analysis

```
 1: function CSP-Analyze-Conflict
 2:     cl:= Explain (conflict-node);
 3:     pred:= Predecessors (conflict-node);
 4:     front:= Relevant (pred, cl);
 5:     while (¬Stop-criterion-met (front)) do
 6:         curr-node:= Last-node (front);
 7:         front:= front\curr-node;
 8:         expl:= Explain (curr-node);
 9:         cl:= Resolve (cl, expl, var(lit(curr-node)));
10:         pred:= Predecessors (curr-node);
11:         front:= Distinct (Relevant (front ∪ pred, cl));
12:     end while
13:     add-clause-to-database(cl);
14:     return clause-asserting-level(cl);
15: end function
```

Example 2.3.2 *Table 2.2 demonstrates a run of Algorithm 2.1 on our running example. Observe that it computes the conflict clause by resolving* ω_3 *with* ω_2*, and the result of this resolution with* ω_1*. The intermediate result, namely the result of the first of these*

resolutions, is discarded. The resulting conflict clause $(a \notin [1,4] \vee b \notin [1,8] \vee c \notin [1,8])$ *is stronger than what the clause would be had we used the 1-UIP method, namely* $(a \neq 1 \vee b \notin [1,8] \vee c \notin [1,8])$.

□

Line	Operation		
2	*cl*	$:= (b \notin [5,8] \vee c \notin [5,8])$	$(= \omega_3)$
3	*pred*	$:= \{b \in [5,8]@1, c \in [5,8]@1\}$	
4	*front*	$:= \{b \in [5,8]@1, c \in [5,8]@1\}$	
6	*curr-node*	$:= c \in [5,8]@1$	
7	*front*	$:= \{b \in [5,8]@1\}$	
8	*expl*	$:= (a \notin [1,4] \vee c \notin [1,4])$	$(= \omega_2)$
9	*cl*	$:= (a \notin [1,4] \vee b \notin [5,8] \vee c \notin [1,8])$	
10	*pred*	$:= \{a = 1@1, c \in [1,8]@0\}$	
11	*front*	$:= \{a = 1@1, b \in [5,8]@1, c \in [1,8]@0\}$	
6	*curr-node*	$:= b \in [5,8]@1$	
7	*front*	$:= \{a = 1@1, c \in [1,8]@0\}$	
8	*expl*	$:= (a \notin [1,4] \vee b \notin [1,4])$	$(= \omega_1)$
9	*cl*	$:= (a \notin [1,4] \vee b \notin [1,8] \vee c \notin [1,8])$	
10	*pred*	$:= \{a = 1@1, b \in [1,8]@0\}$	
11	*front*	$:= \{a = 1@1, b \in [1,8]@0, c \in [1,8]@0\}$	
13	add$((a \notin [1,4] \vee b \notin [1,8] \vee c \notin [1,8]))$		
14	return 0		

Table 2.2: A trace of Algorithm 2.1 on the running example. The horizontal lines separate iterations.

Saving proof data

The resolution steps are saved in a list $s_1, \ldots, s_n$, in case they will be needed for the proof. Each step s_i can be defined by a clause c_i and a resolution variable v_i. The first step s_1 has an undefined resolution variable. The sequence of resolutions is well-defined by this list: the first clause is c_1, and the i-th resolution step for $i \in [2,n]$ is the resolution of c_i with the clause computed in step $i-1$, using v_i as the resolution variable. In practice HCSP refrains from saving explanation clauses owing to space considerations, and instead it infers them again when printing the proof. It represents each proof step with a tuple $\langle Constraint, Rule, Pivot \rangle$, where *Constraint* is a pointer to a constraint in the constraints database, *Rule* is the parameterized inference rule by which an explanation clause can be inferred (if *Constraint* happens to be a clause then

Rule is simply NULL), and *Pivot* is a pointer to the resolution variable. The EXPLAIN function saves this information.

2.3.3 Inferring explanation clauses

We now describe how explanation clauses are generated with the EXPLAIN function.

Every constraint has a propagator, which is an algorithm that deduces new facts. Every such propagator can also be written formally as an inference rule, possibly parameterized. For example, the propagator for a constraint of the form $a \leq b$ when used for inferring a new domain for a, is implemented by computing $\{x \mid x \in D(a) \wedge x \leq \max(D(b))\}$, i.e., by finding the maximal value in the domain of b, and removing values larger than this maximum from $D(a)$. The same deduction can be made by instantiating the inference rule LE(m) below with $m = \max(D(b))$.

$$\frac{a \leq b}{(a \in (-\infty, m] \vee b \in [m+1, \infty))} \quad (\mathrm{LE}(m)) \ .$$

If, for example, the current state is $a \in [1, 10]$ and $b \in [2, 6]$, then the propagator will infer $a \in [1, 6]$. The consequent of LE(6) implies the same literal at the current state, which means that it is an explanation clause. Table 2.3 contains several such inference rules that we implemented in HCSP. In a proof supplement of this article [VS10b] we provide a soundness proof for these rules, and also prove that such an inference rule exists for any constraint.

One way to infer the explanation clauses, then, is to record the inference rule, and its parameter if relevant, by which the literal is inferred during propagation (when progressing to the right on the implication graph). An alternative solution, which is implemented in HCSP, is to derive the inference rules only during conflict analysis, namely when traversing the implication graph from right to left. The reason that this is more efficient is that propagation by instantiation of inference rules is typically more time consuming than direct implementation of the propagator. Hence performance is improved by finding these rules *lazily*, i.e, only when they participate in the learning process and are therefor potentially needed for the proof.

2.4 Deriving a proof of unsatisfiability

If the formula is unsatisfiable, HCSP builds a proof of unsatisfiability, beginning from the empty clause and going backwards recursively. The proof itself is printed in the correct order, i.e., from roots to the empty clause.

Constraint	Parameters	Inf. rule
`AllDifferent`$(v_1, \ldots, v_k)$	Domain D, and a set $V \subseteq \{v_1, \ldots, v_k\}$ such that $1 + \lvert D\rvert = \lvert V\rvert$	$\dfrac{\texttt{AllDifferent}(v_1, \ldots, v_k)}{(\bigvee_{v \in V} v \notin D)}$ $(AD(D, V))$
$a \neq b$	Value m	$\dfrac{a \neq b}{(a \neq m \vee b \neq m)}$ $(\mathrm{Ne}(m))$
$a = b$	Domain D	$\dfrac{a = b}{(a \notin D \vee b \in D)}$ $(\mathrm{Eq}(D))$
$a \leq b + c$	Values m, n	$\dfrac{a \leq b + c}{\begin{array}{ll}(a \in (-\infty, m+n] & \vee \\ b \in [m+1, \infty) & \vee \\ c \in [n+1, \infty) &)\end{array}}$ $(LE_+(m, n))$
$a = b + c$	Values l_b, u_b, l_c, u_c	$\dfrac{a = b + c}{\begin{array}{ll}(a \in [l_b + l_c, u_b + u_c] & \vee \\ b \notin [l_b, u_b] & \vee \\ c \notin [l_c, u_c] &)\end{array}}$ $(EQ^a_+(l_b, u_b, l_c, u_c))$

Table 2.3: Inference rules for some popular constraints, which HCSP uses for generating explanation clauses. The last rule is a *bound consistency* propagation targeted at a.

Recall that with each conflict clause, HCSP saves the series of proof steps $s_1, \ldots, s_k$ that led to it, each of which is a tuple $\langle Constraint, Rule, Pivot \rangle$. We denote by $s_i.Cons$, $s_i.Rule$, and $s_i.Pivot$ these three elements of s_i, respectively.

Algorithm 2.2 receives a conflict clause as an argument — initially the empty clause — and prints its proof. It begins by traversing the proof steps $s_1, \ldots, s_k$ of the conflict-clause. Each such step leads to a recursive call if it corresponds to a conflict-clause that its proof was not yet printed. Next, it checks whether the constraint of each proof step is a clause; if it is not, then it computes its explanation with ApplyRule. This function returns the explanation clause corresponding to the constraint, based on the rule $s_i.Rule$. After obtaining a clause, in lines 18–20 it resolves it with cl, the clause from the previous iteration, and prints this resolution step. Note that the clauses resolved in line 19 can be intermediate clauses that were not made into conflict clauses by the conflict analysis process.

Hence, Algorithm 2.2 prints a signed resolution proof, while adding an inference rule that relates each non-clausal constraint to a clausal consequent, namely the explanation clause.

Example 2.4.1 *First, we need an inference rule for NoOverlap:*

Algorithm 2.2 Printing the proof

```
1: function PrintProof(conflict-clause)
2:     Printed ← Printed ∪ conflict-clause
3:     (s_1, ..., s_k) ←ProofSteps (conflict-clause)
4:     for i ← 1, k do
5:         if s_i.C is a clause and s_i.C ∉ Printed then
6:             PrintProof (s_i.C)
7:         end if
8:     end for
9:     for i ← 1, k do
10:        if s_i.C is a clause then expl ← s_i.C
11:        else
12:            expl ← ApplyRule (s_i.C, s_i.Rule)
13:            Print("Rule:", s_i.Rule)
14:            Print("Premise:", s_i.C, "Consequent:", expl)
15:        end if
16:        if i = 1 then cl ← expl
17:        else
18:            Print("Resolve", cl, expl, "on", s_i.Pivot)
19:            cl ← Resolve(cl, expl, s_i.Pivot)
20:            Print("Consequent:", cl))
21:        end if
22:    end for
23: end function
```

1.	NoOverlap$(b, 4, c, 4)$	premise
2.	$(b \notin [5, 8] \vee c \notin [5, 8])$	1[NO(5,5)]
3.	NoOverlap$(a, 4, c, 4)$	premise
4.	$(a \notin [1, 4] \vee c \notin [1, 4])$	3[NO(1,1)]
5.	$(a \notin [1, 4] \vee b \notin [5, 8] \vee c \notin [1, 8])$	2,4[Resolve(c)]
6.	NoOverlap$(a, 4, b, 4)$	premise
7.	$(a \notin [1, 4] \vee b \notin [1, 4])$	6[NO(1,1)]
8.	$(a \notin [1, 4] \vee b \notin [1, 8] \vee c \notin [1, 8])$	5,7[Resolve(b)]
9.	$(a \notin [5, 8] \vee c \notin [5, 8])$	3[NO(5,5)]
10.	$(a \notin [1, 8] \vee b \notin [1, 8] \vee c \notin [1, 8])$	8,9[Resolve(a)]
11.	$(c \in [1, 8])$	premise
12.	$(a \notin [1, 8] \vee b \notin [1, 8])$	10,11[Resolve(c)]
13.	$(b \in [1, 8])$	premise
14.	$(a \notin [1, 8])$	12,13[Resolve(b)]
15.	$(a \in [1, 8])$	premise
16.	()	14,15[Resolve(a)]

Table 2.4: A deductive proof of the unsatisfiability of the CSP.

$$\frac{\mathit{NoOverlap}(a, l_a, b, l_b)}{(a \notin [m, n + l_b - 1] \vee b \notin [n, m + l_a - 1])} \quad \text{(NO(m,n))} \ ,$$

where m, n *are values such that* $1 - l_b \leq n - m \leq l_a - 1$.

Table 2.4 shows a proof of unsatisfiability of this CSP. This presentation is a beautification of the output of Algorithm 2.2. Note that the length of the proof does not change if interval sizes increase or decrease. For example, $a, b, c \in [1, 80]$ *and* $\mathit{NoOverlap}(a, 40, b, 40)$, $\mathit{NoOverlap}(a, 40, c, 40)$, $\mathit{NoOverlap}(b, 40, c, 40)$, *will require the same number of steps. Also note that the proof does not refer to the variables* x_1, x_2 *and* x_3, *since* HCSP *found an* unsatisfiable core *which does not refer to constraints over these variables.*

2.5 Alternative learning mechanisms

While our focus is on extracting proofs, it is also worth while to compare CSP-Analyze-Conflict to alternatives in terms of the conflict clause that it generates, as it affects both the size of the proof and the performance of the solver.

An alternative to CSP-Analyze-Conflict , recall, is collecting the literals of the 1-UIP. In Example 2.3.2 we saw that 1-UIP results in the weaker conflict clause $(a \neq 1 \vee b \notin [1, 8] \vee c \notin [1, 8])$. After learning this clause the solver backtracks to decision level 0, in which the last two literals are false. At this point the first literal is implied, which removes the value 1 from $D(a)$, giving $D'(a) = [2, 8]$. In contrast, Algorithm 2.1 produces the clause $(a \notin [1, 4] \vee b \notin [1, 8] \vee c \notin [1, 8])$ (see line 13 in Table 2.2). This

clause also causes a backtrack to level 0, and the first literal is implied. But this time the range of values $[1, 4]$ is removed from $D(a)$, giving the smaller domain $D''(a) = [5, 8]$. This example demonstrates the benefit of resolution-based conflict analysis over 1-UIP, and is consistent with the observation made in [LKM03].

Another alternative is the MVS algorithm, which was described in [LKM03] in terms of traversing the assignment stack rather than the implication graph. MVS essentially produces the same conflict clause as Algorithm 2.1, but it assumes that the input formula consists of signed clauses only, and hence does not need explanation clauses. We find Algorithm 2.1 clearer than MVS as its description is much shorter and relies on the implication graph rather than on the assignment stack. Further, it facilitates adoption of well-known SAT techniques and relatively easy development of further optimizations. In [VS10b] we present several such optimizations that allow CSP-Analyze-Conflict to trim more irrelevant graph nodes and learn stronger clauses.

A third alternative is the generalized-nogoods algorithm of EFC [KB05]. There are two main differences between the learning mechanisms:

- EFC generates a separate explanation of each removed *value*. HCSP generates an explanation for each propagation, and hence can remove *sets* of values. This affects not only performance: HCSP's conflict analysis algorithm, unlike EFC's, will work in some cases with infinite domains, e.g., intervals over real numbers.

- EFC generates an explanation eagerly, after each constraint propagation. In contrast HCSP generates an explanation only in response to a conflict, and hence only for constraints that are relevant for generating the conflict clause.

Performance

HCSP performs reasonably well in comparison with state of the art solvers. In the CSC09 competition [DLR09], in the n-ary constraints categories, an early version of HCSP achieved the following results, out of 14 solvers. In the 'extension' subcategory: 6-th in UNSAT, 9-th in SAT, 9-th in total. In the 'intension' subcategory: 1-st in UNSAT, 4-th in SAT, 4-th in total. We intend to publish separately detailed experimental results together with a description of the various optimizations in HCSP.

Chapter 3

Learning general constraints in CSP (long version)

Abstract

We present a new learning scheme for CSP solvers, which is based on learning (general) constraints rather than generalized no-goods or signed-clauses that were used in the past. The new scheme is integrated in a conflict-analysis algorithm reminiscent of a modern systematic SAT solver: it traverses backwards the conflict graph and gradually builds an asserting conflict constraint. This construction is based on new inference rules that are tailored for various pairs of constraints types, e.g., $x \leq y_1 + k_1$ and $x \geq y_2 + k_2$, or $y_1 \leq x$ and $[x, y_2] \not\subseteq [a, b]$. The learned constraint is stronger than what can be learned via signed resolution. Our experiments show clear advantage over the state-of-the-art solver Mistral *in most types of constraints, averaging 25% reduction in fails (time-out or memory-out), 29% reduction in run-time of instances that both engines solved, and 95.9% average reduction in the number of backtracks, when measured on the last CSP competition (2009) benchmarks that include inequality constraints (a total of 2162 benchmarks).*

3.1 Introduction

The ability of CSP solvers to learn new constraints during the solving process possibly shortens run-time by an exponential factor (see, e.g., [KB05]). Despite this fact, and in contrast to SAT solvers, only few CSP solvers use learning, owing to the difficulty of making it cost-effective. Learning in a limited form was present in early CSP solvers, where it was called *nogood learning* [Dec90]. Nogoods are defined as partial assignments

that cannot be extended to a full solution. Later *generalized nogoods* [KB05] (g-nogoods for short) were proposed, which allow *non-assignments* as well, e.g., a g-nogood $(x \nleftarrow 1, y \leftarrow 1)$ means that an assignment in which x is assigned anything but 1 and y is assigned 1 cannot be extended to a solution. This formalism is convenient for representing knowledge obtained by propagators. The g-nogood above, for example, can result from removing 1 from the domain of x, which leads by propagation to removing 1 from the domain of y. G-nogoods may be exponentially stronger than nogoods, as shown in [KB05].

A more general and succinct representation of learned knowledge is in the form of *signed clauses*. Such clauses are disjunctions of *signed literals*, where a signed literal has the form $v \in D$ or $v \notin D$ (called positive and negative signed literals, respectively), where v is a variable and D is a domain of values. Beckert et al. [BHM00b] studied the satisfiability problem of signed CNF, i.e., satisfiability of a conjunction of signed clauses. They proposed an inference system, based on simplification rules and a rule for binary resolution of signed clauses:

$$\frac{(v \in A \vee X) \quad (v \in B \vee Y)}{(v \in (A \cap B) \vee X \vee Y)} [\text{Signed Resolution}(v)] \tag{3.1}$$

where X and Y consist of a disjunction of zero or more literals, A and B are sets of values, and v is called the *pivot* variable. Note that in case v is Boolean and A, B are complementary Boolean domains (e.g., $A = \{0\}, B = \{1\}$) then this rule simplifies to the standard resolution rule for propositional clauses that is used in SAT, namely the consequent becomes $(X \vee Y)$.

As we showed in an earlier publication [VS10a], we used this rule in our CSP solver HCSP [VS10a] (short for HaifaCSP)[1], as part of a general learning scheme based on signed clauses. Using a special inference rule for each type of non-clausal constraint, HCSP inferred a signed clause e that *explains* a propagation by that constraint. This means that e is implied by the constraint, but at the same time is strong enough to make the same propagation as the constraint, at the same state. Using such explanations for propagations by non-clausal constraints, and rule (3.1) for resolving signed clauses, HCSP can generate a signed *conflict clause* via *conflict analysis*. By construction this clause is *asserting* (i.e., it necessarily leads to additional propagation after backtracking). In contrast to the CSP solver EFC [KB05], which generates a g-nogood *eagerly* for each removed *value*, HCSP generates a signed explanation clause *lazily*, only as part of conflict analysis. Lazy learning of g-nogoods was also implemented on top of MINION [GMM10].

In this article we study a different learning scheme, which is based on inference rules with non-clausal consequents. Our main goal in introducing this scheme is to

[1]The early version of our solver that was introduced in [VS10a] was called PCS, for Proof-producing Constraint Solver.

learn a conflict constraint that is logically stronger and easier to compute than its clausal counterpart. The emphasis is on the first of these goals as it may improve the search itself. To that end, we propose a generic inference rule called *Combine* that for many popular (pairs of) constraints indeed fulfills these two goals. For example, suppose that in a state in which the domains of three variables are defined by $x \in \{2, 6, 10, 14, \ldots, 30\}, y_1 \in \{8, 12, 16, 20\}, y_2 \in \{1, 2, 3, \ldots, 9\}$, the constraint $c_1 \doteq y_1 \leq x$ propagates $x \in \{10, 14, \ldots, 30\}$, which leads to a contradiction with a constraint $c_2 \doteq x \leq y_2$. During conflict-analysis, HCSP now infers from this propagation the constraint

$$x \in [8, 9] \vee [y_1, y_2] \not\subseteq [8, 9] \ ,$$

based on *Combine* (square brackets denote a range). This constraint is both implied by c_1, c_2, and has the form of a disjunction of two constraints, each of which has less variables than the set of input variables. The latter property potentially makes it easier to solve. Instantiations of *Combine* always have this property. For some combinations of rules we do not use *Combine* since the result is too complicated to derive or too computationally expensive to support. In such cases we offer simpler alternatives.

Our experimental results prove that indeed the new scheme is better than clausal explanation. For reference, we also compared HCSP to the state-of-the-art CSP solver Mistral [Heb08]. Our experiments with thousands of benchmarks from the latest competition (in 2009), as we describe in Sec. 3.5, show that comparing to Mistral, HCSP achieves an overall 25% reduction in fails (time-out or memory-out), 29% reduction in run-time of instances that both engines solved, and 95.9% average reduction in the number of backtracks. Perhaps this drastic reduction in backtracks indicates that the cost of learning strong constraints is mitigated by a better search (Mistral itself does not learn constraints).

The rest of the article is structured as follows. The next section covers background material, including the learning framework that we use and clausal explanations [VS10a]. Sections 3.3 and 3.4 describe the new set of inference rules, the requirements from them and the proofs that they fulfill these requirements. In Sec. 3.3 we also explain how we use clausal explanations as a fallback solution when we are unable to infer a general constraint that satisfies the required properties. We conclude in Sec. 3.5 with empirical evaluation and some proposals for future research.

3.2 Background

3.2.1 Essentials of HCSP

The engine of HCSP adopts classical ideas from the CSP and SAT literature. We assume the reader is somewhat familiar with those, and only mention several highlights briefly.

HCSP makes a *decision* (variable ordering) by selecting a variable with the highest ratio of *score* to domain-size, where *score* is calculated similarly to Chaff's VSIDS technique [MMZ+01a].[2] The value is initially chosen to be the minimal value in the domain, and after that according to the last assigned value, a technique that is typically referred to by the name *phase saving* is SAT [Sht00]. It includes *restarts*, *learning* (to be described), and *deletion* of learnt-constraints with low activity.

HCSP supports all the constraint types used in the 2009 CSP competition. It has *precise* propagators for the following types of constraints (where x_i denote variables, b_i Boolean variables, a_i constants, and $\diamond \in \{=, \leq, \geq\}$): $x_0 = x_1, x_0 = -x_1$, $x_0 \diamond abs(x_1)$, $x_0 \diamond x_1 + x_2 + \ldots$ (with wrap on overflow), $x_0 \leq x_1 * x_2$, $x_0 \geq x_1 * x_2$, $x_0 = \min(x_1, x_2, \ldots)$, $x_0 = \max(x_1, x_2, \ldots)$, $x_0 = (b_0?x_2 : x_3)$, $x \neq 0 \leftrightarrow y = z$, $x_0 \in Set \leftrightarrow x_1 < x_2$, $x_0 - x_1 \geq a_0$, $(x_0 + a_0 \leq x_1 \vee x_1 + a_1 \geq x_0)$, $x_0 \neq x_1$, AllDifferent$(x_0, x_1, \ldots)$, $[x_0, x_1 + a_0] \subseteq [a_1, a_2]$, $[x_0, x_1 + a_0] \not\subseteq [a_1, a_2]$, $a_0 * x_0 + a_1 * x_1 + ... \geq a_k$, signed-clauses, and a disjunction of any of the above when there are no shared variables. It has *imprecise* propagators for $x_0 = x_1 * x_2$, $x_0 = x_1/x_2$, $x_0 = x_1 \% x_2$, $x_0 = pow(x_1, x_2)$, and table constraints.

Complex constraints modeled by a language such as XCSP [C0909] are rewritten into more basic ones.

The rest of this section is focused on the learning mechanism.

3.2.2 Conflict analysis

An *implication graph* $G(N, E)$ is a directed acyclic graph in which each node $n \in N$ represents a literal (a variable domain) and each edge $c \in E$ represents a constraint. Incoming edges to a node n can only be labeled with the same constraint. Let $(n_1, n), \ldots, (n_k, n)$ be the incoming edges of n, all of which are labeled with a constraint c. This represents the fact that starting with domains $n_1, \ldots, n_k$ the propagator of c inferred the domain in n. The constraint c is called the *antecedent* of n. Each node is also associated with the *decision level* in which the domain reduction occurred.

When an implication graph ends with a conflict (a node labeled with $\perp$), it is called a *conflict graph*. We will follow a convention by which this graph is depicted with the roots at the left and the sink at the right, and the horizontal position of a node indicates

[2] This can be seen as a variant of the *dom/wdeg* strategy [BHLS04].

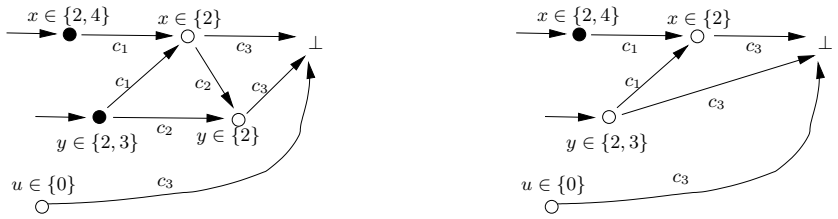


Figure 3.1: Part of a conflict graph, based on the constraints $c_1 \doteq y \geq x$, $c_2 \doteq x \geq y$, and $c_3 \doteq x > y + u$. Empty circles represent nodes in the set F. Drawing (a) is before relaxation, and drawing (b) is right after it. Relaxation discovers that the domain reduction by c_2 is not necessary for conflicting the constraint *curr* (c_3 in this case).

the time it occurred. Examples of conflict graphs can be seen in Fig. 3.1 (the reader is advised to ignore at this stage the distinction between filled and empty nodes in that figure).

HCSP analyzes the conflict graph in order to learn a new constraint, called accordingly a *conflict constraint* (or a conflict *clause* in SAT). A conflict constraint is called *asserting* if there exists a backtrack level in which this constraint necessarily leads to additional propagation. The conflict-analysis function, AnalyzeConflict , indeed computes this level and returns it to the solver, which backtracks accordingly. Alg. 3.1 shows pseudo-code of AnalyzeConflict as implemented in HCSP. It maintains a set of nodes F, which is initialized to the set of nodes that contradict the input constraint *cc*. In line 4 it performs a *relaxation* of F. Relaxation means that each node in F is 'pushed' to the left as long as the constraint *Cons* remains conflicting. Generally this is possible when domain reductions are redundant, as demonstrated in the following example.

Example 3.2.1 *Consider the constraints*

$$c_1 \doteq y \geq x \quad c_2 \doteq x \geq y \quad c_3 \doteq x > y + u \ .$$

and the conflict graph in Fig. 3.1(a). In Alg. 3.1, initially $Cons = c_3$*, and hence after line 2* $F = \{x \in \{2\}, y \in \{2\}, u \in \{0\}\}$ *(those are marked with empty circles). Relaxation in line 4 replaces in* F *the node* $y \in \{2\}$ *with the node* $y \in \{2,3\}$*, because the new* F *also contradicts the current constraint* $Cons$*. Fig. 3.1(b) shows this. The reason that this is possible is that the domain reduction by* c_2 *is redundant in the current state, because when* $u = 0$*,* c_3 *is capable of removing this value by itself. Such cases appear frequently, because the order in which constraints are processed is not optimal.* □

Relaxation is necessary for several reasons:

- Preventing a situation in which the learned clause is still conflicting immediately after backtracking, instead of being asserting,
- In Sec. 3.4.3 we rely on relaxation in the development of some of the rules.
- Our experiments show that without it many more cases fall back to clausal explanations, because relaxation enables to circumvent them.

Relaxation is a contribution of the current article, although not its focus.

Let us return to the description of Alg. 3.1. In lines 5–9 AnalyzeConflict gradually updates the constraint $Cons$. It does so by traversing the conflict graph backwards (i.e., going left, from the conflict node towards the decision node) while updating F and the constraint $Cons$ such that the following loop invariants are maintained:

Invar1. *curr* contradicts the domains defined by F, and is able to detect it via propagation (detection is not a given, because not all constraints have a precise propagator, i.e., they are all sound but not all are complete. Bounds consistency is an example of such imprecise propagation).

Invar2. No two nodes in F refer to the same variable.

It should be clear that these invariants are maintained at the entry to the loop, because of the definition of F, $Cons$, and relaxation. Combine and GetNewSet are targeted towards maintaining it as will be evident later. The traversal stops in line 5 once the function stop detects that $Cons$ is asserting, or that it conflicts the domains at decision level 0. In the latter case the function AssertingLevel returns -1 to the solver, which accordingly declares the CSP to be unsatisfiable. In line 8 the current constraint $Cons$ is replaced with a constraint that is inferred from $Cons$ itself and the antecedent constraint of a node in F. The function combine is the main contribution of this article and will be discussed in length in later sections.

Let us now shift our focus to GetNewSet, which updates the set F. Initially it replaces *pivot* with its parents. In case there is more than one node in F representing the same variable, in line 18 the function distinct leaves only the right-most one. The reason that there may be multiple entries of a variable in F is that a parent of *pivot* may represent a variable that already labels a different node in F because of relaxation (line 11) in a previous iteration.

3.2.3 Clausal Explanations

Generic explanations were used in the past (e.g., [KB05, GMM10]) for learning of g-nogoods. The scheme we describe here uses inference rules specialized for each constraint

Algorithm 3.1 AnalyzeConflict receives as input the currently conflicting constraint, learns a new constraint $Cons$ which is asserting (i.e., necessarily leads to further propagation), and returns the backtrack level. Combine, the subject of Sect. 3.3–3.4, infers a new constraint. GetNewSet computes the new set of nodes F, as explained in the text.

```
1: function AnalyzeConflict (constraint cc)          ▷ cc = conflicting constraint
2:     F ← the set of nodes contradicting cc;
3:     Cons ← cc;
4:     F ←relax (F, Cons);
5:     while !stop (F,Cons) do          ▷ stop if Cons is asserting or UNSAT detected
6:         pivot ← node of F that was propagated last;
7:         antecedent ← incoming constraint of pivot;
8:         Cons ← Combine (Cons, antecedent, pivot, F);
9:         F ← GetNewSet(F, Cons, pivot);
10:        Remove from F nodes referring to variables not in Cons.
11:        F ← relax (F, Cons);          ▷ Go left as long as F contradicts Cons
12:    end while
13:    Add Cons to the constraints database;
14:    return AssertingLevel (Cons,F);    ▷ the backtracking level, or -1 if UNSAT
15: end function

16: function GetNewSet(node-set F, node pivot)
17:    F ← (F \ {pivot}) ∪ parents of pivot;
18:    F ← distinct (F);          ▷ Chooses right-most node of each variable in F
19:    Return F;
20: end function
```

type, resulting in signed clauses. Such clausal explanations are important in our context both for understanding the alternative mechanism that we used in [VS10a] (we use it as one of the points of reference for comparing the results), and because we still use it as a fallback solution when, e.g., we reach pairs of constraints that we do not directly support in COMBINE. This technique is also based on ANALYZECONFLICT , with a difference only in the implementation of COMBINE.

Let us begin by formally defining the notion of explanation.

Definition 3.2.1 (Clausal explanation) *Let $l_1, \ldots, l_n$ be signed literals at the current state (each literal represents the current domain of a variable), and let c be a constraint that propagates the new signed literal l, i.e., $(l_1 \wedge \ldots \wedge l_n \wedge c) \rightarrow l$. Then a clause e is an* explanation *of this propagation if the following two conditions hold:*

$$c \rightarrow e \tag{3.2}$$

$$(l_1 \wedge \cdots \wedge l_n \wedge e) \rightarrow l \ . \tag{3.3}$$

Eq. (3.2) guarantees that the new clause e is logically implied by an existing constraint, hence we do not lose soundness. Eq. (3.3) guarantees that it is still strong enough to imply the same literal. It is always possible to derive an explanation from a constraint, regardless of the constraint type [VS10a].

Example 3.2.2 *The following rule from [VS10a] provides a clausal explanation for an inequality constraint:*

$$\frac{x \leq y}{x \in (-\infty, m] \vee y \in [m+1, \infty)} \quad (LE(m)) \tag{3.4}$$

where m is a parameter instantiating it (the rule is sound for any m). Note that the consequent is a signed clause. Now consider two literals:

$$l_1 \doteq (x \in [1,3]), l_2 \doteq (y \in [0,2])$$

and the constraint

$$c \doteq x \leq y \ ,$$

which implies in the context of l_1, l_2 the literal

$$l \doteq x \in [1,2] \ .$$

Using (3.4) with $m = \max(y) = 2$ we obtain the explanation

$$e \doteq (x \in (-\infty, 2] \vee y \in [3, \infty)) \ ,$$

and indeed (3.2) and (3.3) hold, since $c \rightarrow e$ and $(l_1 \wedge l_2 \wedge e) \rightarrow l$. In [VS10a] alternatives to choosing $m = \max(y)$ are discussed. □

In [VS10a] we showed how HCSP generates a signed conflict clause with an inference system based on signed resolution (3.1), that is reminiscent of how SAT solvers use binary resolution. Explanations are used for bridging between non-clausal constraints and a signed clause (as in the example above), and (3.1) is used for resolving signed clauses.

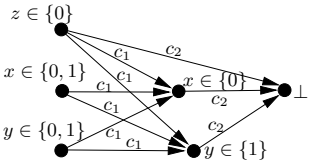
Example 3.2.3 *The following demonstrates conflict analysis with clausal explanations. In addition to (3.4), we will use a variant of this rule for strict inequality:*

$$\frac{x < y}{x \in (-\infty, m-1] \vee y \in [m+1, \infty)} \quad (L(m)) \tag{3.5}$$

We will also use the observation that if $c \to e$, then $(l \vee c) \to (l \vee e)$, to handle constraints with disjunctions. Let $D_x = \{0,1\}$, $D_y = \{0,1\}$, $D_z = \{0..100\}$, and

$$c_1 \doteq (z = 9 \vee x < y) \quad c_2 \doteq (z = 10 \vee x \geq y)\ .$$

The conflict graph on the right shows the decision ($D_z = \{0\}$), and then that c_1 propagates $D_x = \{0\}$, $D_y = \{1\}$ in this order, and finally that c_2 detects a conflict. Now $F = \{z \in \{0\}, x \in \{0\}, y \in \{1\}\}$ and pivot $= \{y \in \{1\}\}$. At this point c_2 generates the explanation



$$(z \in \{10\} \vee x \in [1, \infty) \vee y \in (-\infty, 0])$$

based on LE(0) (Eq. (3.4)), and c_1 generates the explanation

$$(z \in \{9\} \vee y \in [1, \infty) \vee x \in (-\infty, -1])$$

based on L(0) (Eq. (3.5)). Resolving the two explanations on y yields

$$(z \in \{9, 10\} \vee x \notin \{0\}). \tag{3.6}$$

Now pivot $= x \in \{0\}$. c_1 explains the propagation of x with the clause $(z \in \{9\} \vee y \in [2, \infty) \vee x \in (-\infty, 0])$, based on L(1). Resolving it with (3.6) on x yields

$$(z \in \{9, 10\} \vee x \in (\infty, -1] \vee y \in [2, \infty))\ . \tag{3.7}$$

Now F is equal to the three nodes on the left. (3.7) is now asserting, since e.g., at the previous decision level $z \in \{9, 10\}$ is implied. □

3.3 Non-clausal inference: requirements

In Alg. 3.1 COMBINE is given the constraints $Cons(x, \bar{y})$ and $antecedent(x, \bar{y})$ with a joint variable x that appears at the node *pivot*, and some set of variables $\bar{y}$, which may or may not be common to both[3]. It outputs a new constraint over $x, \bar{y}$ that is assigned back into $Cons$. In the presentation that follows we will use $c_1(x, \bar{y})$ to denote $Cons(x, \bar{y})$, $c_2(x, \bar{y})$ to denote $antecedent(x, \bar{y})$, and $c^*(x, \bar{y})$ to denote the output constraint. We also define

$$c_{12}(x, \overline{y}) \doteq c_1(x, \overline{y}) \wedge c_2(x, \overline{y}).$$

Typically we will discard the parameters and write c_1, c_2, c^*, c_{12} instead.

Our first requirement from c^* is that it preserves soundness:

$$c_{12} \rightarrow c^* \ . \tag{3.8}$$

This guarantees that the constraint eventually learned in line 13 is inferred via sound derivations, and hence is guaranteed to be implied by the original CSP.

Let $D'_x, D'_{\overline{y}}$ denote the domains of $x, \overline{y}$ right before the propagation of c_1. Also, let $\vdash_{cp}$ denote the *provability relation* by constraints propagation, i.e., $\phi \vdash_{cp} \psi$ denotes that starting with a set of constraints and domains ϕ, the set of literals ψ is derivable through constraint propagation. Then to preserve *Invar1* (see Sec. 3.2), our second requirement from c^* is:

$$c^*, D'_x, D'_{\overline{y}} \vdash_{cp} \bot \ . \tag{3.9}$$

Finally, we aspire to find the strongest c^* that satisfies the above requirements, and which is easy to propagate.

3.4 Non-clausal inference: rules and their proofs

Rules R1–R6 in Table 3.1 are triples $\langle c_1, c_2, c^* \rangle$ that satisfy the two requirements (3.8) and (3.9). Rules R7 and R8 satisfy (3.8) but not necessarily (3.9). We use them to infer constraints, and then test whether they happen to satisfy (3.9). In addition, we use the following meta-rule for handling disjunctions:

$$\frac{(A \vee c_1) \quad (B \vee c_2)}{A \vee B \vee c^*} \tag{3.10}$$

If $\langle c_1, c_2, c^* \rangle$ satisfies (3.8) and (3.9), then so does (3.10).

[3]It is of course not necessarily the case that they share all the variables, but the description is simplified if we do not consider the shared and unshared variables separately, without sacrificing correctness.

Proof

$$\begin{aligned} & (A \vee c_1) \wedge (B \vee c_2) \\ = \; & (A \wedge B \vee A \wedge c_2 \vee c_1 \wedge B \vee c_1 \wedge c_2) \end{aligned}$$

Clearly

$$A \wedge B \vee A \wedge c_2 \rightarrow A$$

and

$$c_1 \wedge B \rightarrow B$$

and

$$c_1 \wedge c_2 \rightarrow c^*$$

Hence

$$(A \wedge B \vee A \wedge c_2) \vee (c_1 \wedge B) \vee (c_1 \wedge c_2) \rightarrow A \vee B \vee c^*$$

□ □

Example 3.4.1 *We now show two examples in which the rules lead to stronger learning than explanation-based learning*

- *Recall example 3.2.3, which yielded the conflict clause (3.7). Given the same conflict graph but using the meta rule (3.10) with pivot y, we learn instead $z \in \{9, 10\}$, which is clearly stronger.*

- *Consider a variant of the example that was described in the introduction: $x \in \{2, 6, 10, 14, \ldots, 30\}, y_1 \in \{8, 12, 16, 20\}, y_2 \in \{1, 2, 3, \ldots, 9\}$, and constraints*

  $$c_1 \doteq (z \in \{1\} \vee y_1 \leq x) \quad c_2 \doteq (z \in \{1\} \vee x \leq y_2) \; .$$

  Suppose we make a decision $z \in \{0\}$. Then c_1 propagates $x \in \{10, 14, \ldots, 30\}$ and c_2 detects a conflict. Using rule R2 with $k1 = k2 = 0$, and the meta rule (3.10) we obtain:

  $$(z \in \{1\} \vee x \in [8, 9] \vee [y_1, y_2] \not\subseteq [8, 9]) \; . \tag{3.11}$$

  On the other hand if we use explanations, c_2's explanation via LE(9) is $(z \in \{1\} \vee x \in (-\infty, 9] \vee y_2 \in [10, \infty))$, c_1's explanation via LE(7) is

  $$(z \in \{1\} \vee y_1 \in (-\infty, 7] \vee x \in (8, \infty]) \; ,$$

  and resolving these explanations on the pivot x yields

  $$(z \in \{1\} \vee x \in [8, 9] \vee y_1 \in (-\infty, 7] \vee y_2 \in [10, \infty)) \; .$$

  This constraint is strictly weaker than (3.11) because the right disjunct of (3.11) implies $y_1 \leq y_2$.

□

Most of the entries in the table were developed by instantiating a general inference rule called *Combine* (see below), which satisfies these requirements. In some other cases instantiating it turned out to be too complicated and we found c^* without it. Sec. 3.4.3 includes proofs for some of these other rules.

	c_1	c_2	c^*
R1	$x \in X_1 \vee A_1(\overline{y})$	$x \in X_2 \vee A_2(\overline{y})$	$x \in (X_1 \cap X_2) \ \vee \ A_1(\overline{y}) \vee A_2(\overline{y})$
R2	$y_1 \leq x - k_1$	$x \leq y_2 - k_2$	$(x \in \left[k_1 + \min(D'_{y_1}), \max(D'_{y_2}) - k_2\right]) \ \vee$ $([y_1, y_2 - k_2 - k_1] \not\subseteq [\min(D'_{y_1}), \max(D'_{y_2}) - k_2 - k_1])$
R3	$y_1 \leq x$	$[x, y_2] \not\subseteq [a, b]$	$(a > x \geq \min(D'_{y_1})) \vee$ $([y_1, y_2] \not\subseteq [\min(D'_{y_1}), b])$
R4	$x \leq y_1 - k_1$	$[y_2, x - k_2] \not\subseteq [a, b]$	$(\max(D'_{y_1})\text{-}k_1 \geq x > b\text{+}k_2) \vee$ $[y_2, y_1\text{-}k_1\text{-}k_2] \not\subseteq [a, \max(b, \max(D'_{y_1}\text{-}k_1\text{-}k_2))]$
R5	$[y_1, x] \not\subseteq [a_1, b_1]$	$[x, y_2] \not\subseteq [a_2, b_2]$	$(a_2 > x > b_1) \ \vee \ ([y_1, y_2] \not\subseteq [a_1, b_2])$
R6	$[x, y] \not\subseteq [a_1, b_1]$	$[y, x] \not\subseteq [a_2, b_2]$	$(x \in (D'_y \setminus ([a_1, b_1] \cup [a_2, b_2])) \vee$ $y \notin (D'_y \cup [a_1, b_1] \cup [a_2, b_2]))$
R7	$y \leq x + k_1$	$x \leq y + k_2$	$\begin{cases} -k_1 \leq x - y \leq k_2 & \text{if } k_1 + k_2 \geq 0 \\ \bot & \text{otherwise} \end{cases}$
R8	$ax+$ $\sum_{i=1}^{n} a_i y_i \geq k_1$	$-ax+$ $\sum_{i=1}^{n} b_i y_i \geq k_2$	$\sum_{i=1}^{n} (a_i + b_i) x_i \geq k_1 + k_2$

Table 3.1: Triples $\langle c_1, c_2, c^* \rangle$ that we use for deriving conflict constraints. The top part include rules that satisfy both (3.8) and (3.9), whereas the others are only guaranteed to satisfy (3.8). When using them we *test* if they satisfy (3.9). Combinations of a pair of linear constraints can be brought to the form expected by R8 if the coefficients of x have opposite signs, via multiplication by a positive constant. Note that the coefficients a_i, b_i may be 0 in R8.

Since not all combinations of rule types are supported, not all propagators are precise (i.e., logically complete) and not all rules are precise (see R7, R8 in the table), then COMBINE uses explanation-based inference (see Sec. 3.2.3) as a fallback solution. Pseudocode of COMBINE, which is rather self-explanatory, appears in Alg. 3.2.

Algorithm 3.2 COMBINE infers a new constraint c^* from c_1, c_2, which satisfies (3.8) and (3.9), the requirements listed in Sec. 3.3.

1.-1: **function** COMBINE(constraint c_1, constraint c_2, node *pivot*, node-set F)
2.-1: $F' =$ GETNEWSET $(F, pivot)$;
3.-1: **if** the combination of c_1, c_2 is supported **then**
4.-1: $con =$ infer(c_1,c_2, *pivot*); ▷ One of the rules in Table 3.1.
5.-1: **if** $F', con \vdash_{cp} \bot$ **then return** *con*; ▷ *con* satisfies *Invar1*
6.-1: **end if**
7.-1: **end if**
8.-1: $e_1 \leftarrow explain(c_1, parents(pivot), pivot)$; ▷ Fallback: use explanations.
9.-1: $e_2 \leftarrow explain(c_2, F, \bot)$;
10.-1: **return** resolve(e_1,e_2,*pivot*); ▷ Signed resolution
11.-1: **end function**

3.4.1 A generic inference rule: *Combine*

Let S be some set of values. Then it is not hard to see that the following is a contradiction for any constraint $c(x, \overline{y})$:

$$c(x, \overline{y}) \wedge x \in S \wedge \forall x' \in S.\ \neg c(x', \overline{y})\ , \tag{3.12}$$

or, equivalently, that the following implication is valid:

$$c(x, \overline{y}) \rightarrow (x \notin S \vee \exists x' \in S.\ c(x', \overline{y}))\ . \tag{3.13}$$

Let $\mathcal{X}$ denote the set of values of x which have no support in $D'_{\overline{y}}$:

$$\mathcal{X} = \{x' \mid \forall \overline{y'} \in D'_{\overline{y}}\ .\ \neg c_{12}(x', \overline{y'})\}\ . \tag{3.14}$$

Instantiating (3.13) with c_{12} for c and with $\mathcal{X}$ for S yields the inference rule that we call *Combine*:

$$\frac{c_{12}(x, \overline{y})}{(x \notin \mathcal{X} \vee \exists x' \in \mathcal{X}.c_{12}(x', \overline{y}))}\ (Combine) \tag{3.15}$$

Since (3.15) is just an instantiation of (3.13), then (3.15) is clearly sound, and hence (3.8) is satisfied. To satisfy (3.9) we first prove logical entailment ($\models$), which is weaker than the requirement of (3.9) for provability ($\vdash_{cp}$).

Lemma 3.4.1 $c^*, D'_x, D'_{\overline{y}} \models \bot$.

Proof In our case $c^* \doteq (x \notin \mathcal{X} \vee \exists x' \in \mathcal{X}.c_{12}(x', \overline{y}))$. Falsely assume that c^* is satisfied for an assignment of values $a \in D'_x, \overline{b} \in D'_{\overline{y}}$ to $x, \overline{y}$, respectively. Consider the two disjuncts of c^*:

- Suppose $x \notin \mathcal{X}$ is satisfied. Considering the definition of $\mathcal{X}$ in (3.14), this implies that a is supported in c_{12}, or formally

$$\exists \overline{y'} \in D'_{\overline{y}}.\ c_{12}(a, \overline{y'})\ . \tag{3.16}$$

  Based on *Invar1* we know that $c_{12}(x, \overline{y}), D'_x, D'_{\overline{y}} \models \bot$, and hence $\forall x \in D'_x \neg \exists \overline{y} \in D'_{\overline{y}}.\ c_{12}(x, \overline{y})$, and particularly for $x = a$, $\neg \exists \overline{y} \in D'_{\overline{y}}.\ c_{12}(a, \overline{y})$, which contradicts (3.16).

- Now suppose $\exists x' \in \mathcal{X}.\ c_{12}(x', \overline{y})$ is satisfied. Expanding $\mathcal{X}$ and substituting $\overline{y}$ with its assignment $\overline{b}$ yields

$$\exists x'.\ \forall \overline{y'} \in D'_{\overline{y}}.\ \neg c_{12}(x', \overline{y'}) \wedge c_{12}(x', \overline{b})\ .$$

  Since $\overline{b} \in D'_{\overline{y}}$ and $\neg c_{12}(x', \overline{y'})$ is satisfied for all $\overline{y'} \in D'_{\overline{y}}$, then it is satisfied for $\overline{y'} = \overline{b}$. This implies a contradiction: $\exists x'.\ \neg c_{12}(x', \overline{b}) \wedge c_{12}(x', \overline{b})$.

Hence, $x \in D'_x, \overline{y} \in D'_{\overline{y}}$ falsifies c^*, which completes our proof. □ □ It is trivial to see that this lemma implies (3.9) when $\vdash_{cp}$ is precise constraint propagation. When imprecise propagation is involved, e.g., $\vdash_{cp}$ is defined by bounds consistency [CHLS06], HCSP checks whether the constraint happens to be conflicting, and if not it falls back to clausal explanation.

The relative strength of *Combine*.

Two observations about the strength of *Combine* that we prove below are:

- There is no alternative to $\mathcal{X}$ for replacing S in (3.13) that makes the resulting constraint stronger, and
- The signed clause that we obtain through the explanation mechanism—see Sec. 3.2.3—cannot yield a stronger consequent.

Lemma 3.4.2 *Consider all possible formulas of the form $\psi(x, \overline{y}) \equiv (x \notin P \vee \varphi(\overline{y}))$, for a given set P. The strongest possible $\varphi(\overline{y})$, which meets all the requirements is $\exists x' \in P.c_{12}(x', \overline{y})$. In other words, for any $\varphi(\overline{y})$ which makes $\psi(x, \overline{y})$ meet the requirements, the following is satisfied:*

$$\exists x' \in P.c_{12}(x', \overline{y}) \models \varphi(\overline{y})$$

Proof By negation, assume that there is and assignment $\overline{b}$ to $\overline{y}$ which satisfies $\exists x' \in P.c_{12}(x', \overline{b})$ but not $\varphi(\overline{b})$. This means that there is an $a \in P$ which satisfies $c_{12}(a, \overline{b})$ when $\neg\varphi(\overline{b})$. Because $c_{12}(a, \overline{b})$ is satisfied, (3.8) mandate that $\psi(a, \overline{b})$ should also be satisfied. According to the definition of ψ we conclude that either $a \notin P$ or $\varphi(\overline{b})$ have to be satisfied. But since $\neg\varphi(\overline{b})$ and, as defined, $a \in P$ we conclude that ψ is unsatisfied with $a, \overline{b}$. This conflict the initial assumption. This leads to the conclusion that if $\exists x' \in P.c_{12}(x', \overline{y})$ is satisfied, then $\varphi(\overline{y})$ must also be satisfied. □ Note that this lemma refers to P, and not $\mathcal{X}$. This means that it does not rule out the possibility where $\exists x' \in P.c_{12}(x', \overline{y})$ is stronger than $\exists x' \in \mathcal{X}.c_{12}(x', \overline{y})$. It is quite possible that the smaller the set P is and the weaker the literal $x \notin P$ is, the stronger $\exists x' \in P.c_{12}(x', \overline{y})$ becomes.

Lemma 3.4.3 *Consider all possible* $\psi(x, \overline{y}) \equiv (x \notin P \vee \exists x' \in P.c_{12}(\overline{y}))$ *which satisfy the requirements.* P *must satisfy*

$$D'_x \subseteq P \ .$$

Proof By negation assume that $D'_x \not\subseteq P$, i.e., there is a value a such that $a \in D'_x$ and $a \notin P$. Consider how this affects (3.9). According to (3.9),

$$x \notin P \vee \exists x' \in P.c_{12}(x', \overline{y}), D'_x, D'_{\overline{y}} \vdash_{cp} \bot \ .$$

Since $a \in D'_x$ then we can replace x with a in the above formula, and get

$$a \notin P \vee \exists x' \in P.c_{12}(x', \overline{y}), D'_{\overline{y}} \vdash_{cp} \bot \ .$$

But since a was defined such that $a \notin P$ then the above formula becomes

$$\mathit{true} \vee \exists x' \in P.c_{12}(x', \overline{y}), D'_{\overline{y}} \vdash_{cp} \bot \ .$$

This basically says $\mathit{true} \vdash_{cp} \bot$, which is impossible. It implies that the assumption that $D'_x \not\subseteq P$ is incorrect. □ □

Lemma 3.4.4 *Consider all possible* $\psi(x, \overline{y}) \equiv (x \notin P \vee \exists x' \in P.c_{12}(\overline{y}))$ *which satisfy the requirements.* P *must satisfy*

$$P \subseteq \mathcal{X} \ .$$

Where $\mathcal{X}$ *was defined above as*

$$\mathcal{X} = \left\{ x' | \forall \overline{y'} \in D'_{\overline{y}}. \left[\neg c_{12}(x', \overline{y'}) \right] \right\} \ .$$

Proof According to the definition of $\mathcal{X}$, the lemma can be reformulated as

$$\forall a \in P \forall \overline{y'} \in D'_{\overline{y}}. \left[\neg c_{12}(a, \overline{y'})\right] \ .$$

Assume, by negation, that this is not correct. In other words there are $a \in P$ and $\overline{b} \in D'_{\overline{y}}$ such that $c_{12}(a, \overline{b})$. We will show that this conflicts (3.9).

Due to (3.9) we know that

$$x \notin P \vee \exists x' \in P. c_{12}(x', \overline{y}), D'_x, D'_{\overline{y}} \vdash_{cp} \bot \ .$$

Since $\overline{b} \in D'_{\overline{y}}$, the formula above implies

$$x \notin P \vee \exists x' \in P. c_{12}(x', \overline{b}), D'_x \vdash_{cp} \bot \ .$$

Assuming $\vdash_{cp}$ is precise this implies

$$\forall x \in D'_x. \neg \left[x \notin P \vee \exists x' \in P. c_{12}(x', \overline{b})\right] \ .$$

We now push the negation down, and get

$$\forall x \in D'_x. \left[x \in P \wedge \forall x' \in P. \neg c_{12}(x', \overline{b})\right] \ .$$

First, we see that this implies $D'_x \subseteq P$. Since x is independent in the formula we conclude that

$$\forall x' \in P. \neg c_{12}(x', \overline{b}) \ .$$

Now we go back to $a \in P$ and $\overline{b} \in D'_{\overline{y}}$ which guarantee $c_{12}(a, \overline{b})$, and combine it with the formula above. This means that we can assign $x' = a$, which leads to $\neg c_{12}(a, \overline{b})$, which conflicts with the guarantee of $c_{12}(a, \overline{b})$. This means that our assumption that the lemma is incorrect was wrong, hence $P \subseteq \mathcal{X}$. $\square$ $\square$ This lemma implies that the literal $x \notin \mathcal{X}$ is the strongest possible. This does not imply anything regarding the strength of the second part of the formula, i.e., $\exists x' \in \mathcal{X}. [c_{12}(x', \overline{y})]$ may be weakened by strengthening $x \notin P$.

Also note that the previous lemmas bound P to $D'_x \subseteq P \subseteq \mathcal{X}$.

Theorem 3.4.1 *There is no alternative $\psi(x, \overline{y})$, different than c^*, which is stronger than c^* with $\mathcal{X}$. In other words*

$$\psi(x, \overline{y}) \not\equiv [x \notin \mathcal{X} \vee \exists x' \in \mathcal{X}. c_{12}(x', \overline{y})]$$

$$\Downarrow$$

$$(\psi(x, \overline{y}) \not\models [x \notin \mathcal{X} \vee \exists x' \in \mathcal{X}. c_{12}(x', \overline{y})]) \ .$$

Proof Recall that we require that $\psi(x, \overline{y})$ to be of the form $x \notin P \vee \varphi(\overline{y})$. Lemma 3.4.2 shows that for a given P, the strongest possible $\varphi(\overline{y})$ is $\exists x' \in P.c_{12}(x', \overline{y})$. This leaves us to prove that

$$x \notin P \vee \exists x' \in P.c_{12}(x', \overline{y})$$

is not stronger than

$$x \notin \mathcal{X} \vee \exists x' \in \mathcal{X}.c_{12}(x', \overline{y}) \ .$$

According to Lemma 3.4.4 because P complies with the given requirements then $P \subseteq \mathcal{X}$. If $P = \mathcal{X}$ the two formulas are equivalent and neither are stronger, otherwise $P \subset \mathcal{X}$.

Assume that $P \subset \mathcal{X}$, this means that there is a such that $a \in \mathcal{X}$ and $a \notin$P. As a result, for $x = a$ the literal $x \notin P$ is true and the literal $x \notin \mathcal{X}$ is false. In this situation the P based formula, i.e., $x \notin P \vee \exists x' \in P.c_{12}(x', \overline{y})$ is true, but the $\mathcal{X}$ based formula depends solely on

$$\exists x' \in \mathcal{X}.c_{12}(x', \overline{y}) \ .$$

We look for a case where this formula is falsified when $x = a$. It can be falsified, i.e., not a tautology, since otherwise this would conflict (3.9). This means that there is an assignment $\overline{b}$ to $\overline{y}$ such that the formula is falsified. We have found $x = a$ and $y = \overline{b}$ for which the P based formula is satisfied and the $\mathcal{X}$ based formula is falsified. This means that the P based formula is not stronger than the $\mathcal{X}$ based formula.

Because the P based formula is the strongest possible form of $x \notin P \vee \varphi(\overline{y})$, this implies that any $\psi(x, \overline{y})$ that satisfies the requirements is not stronger than

$$x \notin \mathcal{X} \vee \exists x' \in \mathcal{X}.c_{12}(x, \overline{y}) \ .$$

□ □ Note that this theorem does not say that, with $\mathcal{X}$, the resulting constraint is stronger than any other possibility; it says that no other constraint is stronger. In other words, there need not be a strict ordering of constraints.

3.4.2 Selected rules based on instantiating *Combine*

We now instantiate *Combine* (3.15) with several specific constraints of interest. The derivations rely on various properties of the domains before propagation $D'_x, D'_{\overline{y}}$ and right after it $D''_x, D''_{\overline{y}}$. By definition

$$c_1, D'_x, D'_{\overline{y}} \vdash_{cp} (x \in D''_x \wedge y \in D''_{\overline{y}}) \ . \tag{3.17}$$

We make the following observations about these domains:

1. The domain of x, and possibly domains of variables in $\bar{y}$, are reduced by c_1:

$$D''_x \subset D'_x, \quad D''_{\overline{y}} \subseteq D'_{\overline{y}} \ . \tag{3.18}$$

2. Owing to *Invar1*, in the context of $D''_x, D''_{\overline{y}}$, c_2 detects a conflict:

$$c_2(x, \overline{y}), D''_x, D''_{\overline{y}} \vdash_{cp} \bot \ . \tag{3.19}$$

3. c_1 cannot detect a conflict on its own in the context of $D'_x, D'_{\overline{y}}$:

$$c_1, D'_x, D'_{\overline{y}} \nvdash_{cp} \bot \ . \tag{3.20}$$

We now use these observations when instantiating *Combine*.

Rule R1: $\quad c_1 \doteq x \in X_1 \vee A_1(\overline{y}) \qquad c_2 \doteq x \in X_2 \vee A_2(\overline{y})$

A_1 and A_2 are disjunctions of zero or more literals over the variables of $\overline{y}$. Expanding c_{12} in (3.14) yields

$$\mathcal{X} = \{x' \mid \forall \overline{y'} \in D'_{\overline{y}}.\ (x' \notin X_1 \wedge \neg A_1(\overline{y})) \vee (x' \notin X_2 \wedge \neg A_2(\overline{y}))\} \ .$$

From (3.17) and (3.18) we know that $c_1(x, \overline{y}), D'_x, D'_{\overline{y}} \vdash_{cp} x \in D''_x$ and $D'_x \neq D''_x$, which implies that $A_1(\overline{y}), D'_{\overline{y}} \models \bot$, and consequently simplifies the above to

$$\mathcal{X} = \{x' \mid \forall \overline{y'} \in D'_{\overline{y}}.\ x' \notin X_1 \vee (x' \notin X_2 \wedge \neg A_2(\overline{y}))\} \ .$$

Note that the propagation of c_1 in the context of $D'_{\overline{y}}, D'_x$ results in $D''_x = X_1 \cap D'_x$ and $D'_{\overline{y}} = D''_{\overline{y}}$

Since $D'_{\overline{y}} = D''_{\overline{y}}$ and, according to (3.19), $c_2(x, \overline{y}), D''_x, D''_{\overline{y}} \vdash_{cp} \bot$, then $A_2(\overline{y}), D'_{\overline{y}} \models \bot$. This means that $\forall \overline{y'} \in D'_{\overline{y}}.\neg A_2(\overline{y})$, which simplifies the above formula to

$$\mathcal{X} = \{x' \mid \forall \overline{y'} \in D'_{\overline{y}}.\ x' \notin X_1 \vee x' \notin X_2\} \ .$$

Since the inner part does not depend on $\overline{y}$ the formula is further simplified to

$$\mathcal{X} = \{x' \mid x' \notin X_1 \vee x' \notin X_2\} \ .$$

Using this definition of $\mathcal{X}$ we examine c^*:

$$c^*(x, D'_{\overline{y}}) = (x \in (X_1 \cap X_2) \vee \exists x' \notin (X_1 \cap X_2).\ c_{12}(x', \overline{y})) \ .$$

Let $A' = \exists x' \notin (X_1 \cap X_2).\ c_{12}(x', \overline{y})$. This simplifies the above to

$$c^*(x, D'_{\overline{y}}) = (x \in (X_1 \cap X_2) \vee A') \ . \tag{3.21}$$

We split the quantifier in A' into three cases:

$$\begin{aligned} A' = \big(&\exists x' \in (X_1 \setminus X_2).\ c_{12}(x', \overline{y}) \quad \vee \\ &\exists x' \in (X_2 \setminus X_1).\ c_{12}(x', \overline{y}) \quad \vee \\ &\exists x' \notin (X_2 \cup X_1).\ c_{12}(x', \overline{y})\big) \qquad . \end{aligned}$$

After taking the definitions of c_{12}, c_1, and c_2 into account:

$$\begin{aligned} A' = \big(&\exists x' \in (X_1 \setminus X_2).\ A_2(\overline{y}) \quad \vee \\ &\exists x' \in (X_2 \setminus X_1).\ A_1(\overline{y}) \quad \vee \\ &\exists x' \notin (X_2 \cup X_1).\ A_1(\overline{y}) \wedge A_2(\overline{y})\big) \qquad . \end{aligned}$$

Next, we eliminate the $\exists$ quantifier and get

$$\begin{aligned} A' = \big(&(X_1 \setminus X_2 \neq \emptyset \wedge A_2(\overline{y})) \quad \vee \\ &(X_2 \setminus X_1 \neq \emptyset \wedge A_1(\overline{y})) \quad \vee \\ &((X_2 \cup X_1)^C \neq \emptyset \wedge A_1(\overline{y}) \wedge A_2(\overline{y}))\big) \qquad . \end{aligned}$$

We simplify this further by showing that $X_1 \setminus X_2 \neq \emptyset$ and $X_2 \setminus X_1 \neq \emptyset$, which leads to $A' = (A_1(\overline{y}) \vee A_2(\overline{y}))$.

According to (3.17),(3.18), $(x \in X_1 \vee A_1(\overline{y})), D'_x, D'_{\overline{y}} \models x \in D''_x$, where $D''_x \subset D'_x$, which implies that $A_1(\overline{y}), D'_{\overline{y}} \models \bot$. Similarly $(x \in X_1 \vee A_1(\overline{y})), D''_x, D'_{\overline{y}} \models \bot$ implies that $A_2(\overline{y}), D'_{\overline{y}} \models \bot$. These facts show that A_1 and A_2 are falsified in this context, meaning that we can focus only on $x \in X_1$ and $x \in X_2$ parts of c_1 and c_2 there.

For the following we assume that $\vdash_{cp}$ for signed-clauses is precise, i.e., $\psi \models \phi$ iff $\psi \vdash_{cp} \phi$.

- $X_2 \setminus X_1 \neq \emptyset$. Because $c_2(x, \overline{y}), D'_x, D'_{\overline{y}} \nvdash_{cp} \bot$ and $\vdash_{cp}$ is assumed to be precise then there is an assignment $a \in D'_x$, $\overline{b} \in D'_{\overline{y}}$ such that $c_2(a, \overline{b})$ is satisfied. Since $A_2(\overline{y}), D'_{\overline{y}} \models \bot$ and $\overline{b} \in D'_{\overline{y}}$ we know that $A_2(\overline{b}) \models \bot$, which implies that $a \in X_2$. Further, since $(c_1(x, \overline{y}) \wedge c_2(x, \overline{y})), D'_x, D'_{\overline{y}} \models \bot$ and $c_2(a, \overline{b})$ is satisfied then $c_1(a, \overline{b}) \models \bot$, i.e., $(a \in X_1 \vee A_1(\overline{b})), D'_{\overline{y}} \models \bot$. This leads to $a \notin X_1$, which together with $c \in X_2$ implies $X_2 \setminus X_1 \neq \emptyset$.

- $X_1 \setminus X_2 \neq \emptyset$. Because $c_1(x, \overline{y}), D'_x, D'_{\overline{y}} \not\models \bot$ and since $A_1(\overline{y}), D'_{\overline{y}} \models \bot$ we know that there is at least one element $a \in D'_x$ such that $a \in X_1$. Again, since $(c_1(a, \overline{y}) \wedge c_2(a, \overline{y})), D'_{\overline{y}} \models \bot$ and $c_1(a, \overline{y}), D'_{\overline{y}} \not\models \bot$ then we know that $c_2(a, \overline{y}), D'_{\overline{y}} \models \bot$. This implies that $a \in X_2 \models \bot$, i.e., $a \notin X_2$. Since $a \notin X_2$ and $a \in X_1$ then it follows that $X_1 \setminus X_2 \neq \emptyset$.

These two facts simplify A' to

$$A' = (A_2(\overline{y}) \vee A_1(\overline{y})) \ ,$$

and correspondingly, according to (3.21),

$$c^*(x, \overline{y}) = (x \in (X_1 \cap X_2) \vee A_1(\overline{y}) \vee A_2(\overline{y})) \ . \tag{3.22}$$

Note the equivalence of (3.22) and the result of signed resolution in (3.1).

Rule R2: $\quad c_1 \doteq y_2 - x \geq k_2 \qquad c_2 \doteq x - y_1 \geq k_1$

Expanding c_{12} in (3.14) yields

$$\begin{aligned} \mathcal{X} &= \left\{x' \mid \forall \overline{y'} \in D'_{\overline{y}}.\ [y_2 - x < k_2 \ \vee \ x - y_1 < k_1]\right\} \\ &= \left\{x' \mid \max(D'_{y_2}) - x < k_2 \ \vee \ x - \min(D'_{y_1}) < k_1\right\} \\ &= \left\{x' \mid \max(D'_{y_2}) - k_2 < x \ \vee \ x < k_1 + \min(D'_{y_1})\right\} \ . \end{aligned}$$

The complement of $\mathcal{X}$ can be written as

$$\mathcal{X}^c = \left[k_1 + \min(D'_{y_1}),\ \max(D'_{y_2}) - k_2\right] \ . \tag{3.23}$$

Recall (3.14): $x \notin \mathcal{X} \vee \exists x' \in \mathcal{X}.\ c_{12}(x', \overline{y})$. The right disjunct is equal to:

$$\begin{aligned} & \exists x'.\ x' \in \mathcal{X} \wedge [y_2 - x' \geq k_2 \wedge x' - y_1 \geq k_1] \\ = \ & \exists x'.\ x' \in \mathcal{X} \wedge [y_2 - k_2 \geq x' \geq y_1 + k_1] \\ = \ & \exists x'.\ x' \in \mathcal{X} \wedge x' \in [y_1 + k_1, y_2 - k_2] \ . \end{aligned} \tag{3.24}$$

We use (3.23) to rewrite (3.24):

$$\exists x'.\ x' \notin \left[k_1 + \min(D'_{y_1}),\ \max(D'_{y_2}) - k_2\right] \wedge x' \in [y_1 + k_1, y_2 - k_2] \ ,$$

which implies

$$\begin{aligned} & [y_1 + k_1, y_2 - k_2] \not\subseteq \left[k_1 + \min(D'_{y_1}),\ \max(D'_{y_2}) - k_2\right] \\ = \ & [y_1, y_2 - k_2 - k_1] \not\subseteq \left[\min(D'_{y_1}),\ \max(D'_{y_2}) - k_2 - k_1\right] \ . \end{aligned}$$

Hence, the rule is

$$\frac{y_2 - x \geq k_2 \qquad x - y_1 \geq k_1}{\begin{array}{l} \big(x \in \left[k_1 + \min(D'_{y_1}),\ \max(D'_{y_2}) - k_2\right] \ \vee \\ [y_1,\ y_2 - k_2 - k_1] \not\subseteq \left[\min(D'_{y_1}),\ \max(D'_{y_2}) - k_2 - k_1\right] \big) \end{array}} \tag{3.25}$$

Rule R4: $c_1 \doteq y_1 - x \geq k_1 \qquad c_2 \doteq [y_2, x - k_2] \not\subseteq [a_2, b_2]$

Note that R2 is implied by c_1, c_2. If $y_1 - x \geq k_1$ and $x - y_2 \geq k_2$ conflict D'_x and $D'_{\overline{y}}$ then we can simply use R2. This allows us to concentrate on the case where propagating$[y_2, x - k_2] \not\subseteq [a_2, b_2]$ D'_x and $D'_{\overline{y}}$ removes at least one value from the domain x more than $x - y_2 \geq k_2$. The extra value removed, depicted γ, should satisfy

$$\left(\exists y_2 \in D'_{y_2}.\gamma - y_2 \geq k_2\right) \wedge \forall y_2 \in D'_{y_2}.\ [y_2, \gamma - k_2] \subseteq [a_2, b_2]\ .$$

This is simplified to

$$\gamma - \min(D'_{y_2}) \geq k_2 \wedge [\min(D'_{y_2}), \gamma - k_2] \subseteq [a_2, b_2]$$

From this we conclude, according to the definition of interval inclusion, that

$$\min(D'_{y_2}) \leq \gamma - k_2 \wedge \left(\min(D'_{y_2}) > \gamma - k_2 \vee (a_2 \leq \min(D'_{y_2}) \leq \gamma - k_2 \leq b_2)\right)\ .$$

Since $\min(D'_{y_2}) \leq \gamma - k_2$ and $\min(D'_{y_2}) > \gamma - k_2$ contradict it is possible to eliminate the disjunct $\min(D'_{y_2}) > \gamma - k_2$ and get

$$\min(D'_{y_2}) \leq \gamma - k_2 \wedge a_2 \leq \min(D'_{y_2}) \leq \gamma - k_2 \leq b_2\ .$$

In which we have two copies of $\min(D'_{y_2}) \leq \gamma - k_2$ which one of them can be eliminated

$$a_2 \leq \min(D'_{y_2}) \leq \gamma - k_2 \leq b_2\ .$$

Subsequently

$$a_2 \leq \min(D'_{y_2}) \leq b_2. \tag{3.26}$$

The next step is to find the value of $\mathcal{X}$.

$$\mathcal{X} = \left\{x' | \forall \overline{y'} \in D'_{\overline{y}}.\neg c_{12}(x', \overline{y'})\right\}\ .$$

we get

$$\mathcal{X} = \left\{x' | \forall \overline{y'} \in D'_{\overline{y}}.\ [y_1 - x < k_1 \vee [y_2, x - k_2] \subseteq [a_2, b_2]]\right\}\ .$$

Considering the properties of $\subseteq$ and of $<$ we replace y_1 with $\max(D'_{y_1})$ and and y_2 with $\min(D'_{y_2})$:

$$\mathcal{X} = \left\{x | \max(D'_{y_1}) - x < k_1 \vee \left[\min(D'_{y_2}), x - k_2\right] \subseteq [a_2, b_2]\right\}\ .$$

When we expand according to the definition of $\subseteq$ we get:

$$\mathcal{X} = \left\{x | \max(D'_{y_1}) - x < k_1 \vee \left(\min(D'_{y_2}) > x - k_2 \vee (\min(D'_{y_2}) \geq a_2 \wedge x - k_2 \leq b_2)\right)\right\}\ .$$

We apply $\min(D'_{y_2}) \geq a_2$, from Eq.(3.26), on the above equation

$$\mathcal{X} = \left\{x | \max(D'_{y_1}) - x < k_1 \vee \left(\min(D'_{y_2}) > x - k_2 \vee x - k_2 \leq b_2\right)\right\} \ .$$

We also apply $b_2 \geq \min(D'_{y_2})$, from Eq.(3.26), on the above equation

$$\mathcal{X} = \left\{x | \max(D'_{y_1}) - x < k_1 \vee x - k_2 \leq b_2\right\} \ .$$

Normalizing x gives

$$\mathcal{X} = \left\{x | \max(D'_{y_1}) - k_1 < x \vee x \leq b_2 + k_2\right\} \ .$$

In terms of intervals, this can be written as

$$\mathcal{X} = (b_2 + k_2, \max(D'_{y_1}) - k_1]^C \quad ,$$

and if we assume that all values are integers then

$$\mathcal{X} = [b_2 + k_2 + 1, \max(D'_{y_1}) - k_1]^C \ .$$

$$c^* \doteq x \notin \mathcal{X} \vee \exists x' \in \mathcal{X}. \, (y_1 - x' \geq k_1 \wedge [y_2, x' - k_2] \not\subseteq [a_2, b_2]) \ .$$

We explore the right disjunct

$$\exists x' \in \mathcal{X}. \, (y_1 - k_1 \geq x' \wedge [y_2, x' - k_2] \not\subseteq [a_2, b_2]) \ .$$

Applying $\mathcal{X}$ we get

$$\exists x'. \left[\max(D'_{y_1}) - k_1 < x' \wedge y_1 - x' \geq k_1 \wedge [y_2, x' - k_2] \not\subseteq [a_2, b_2]\right] \vee$$

$$\exists x'. \, [x' \leq b_2 + k_2 \wedge y_1 - x' \geq k_1 \wedge [y_2, x' - k_2] \not\subseteq [a_2, b_2]] \ .$$

Expanding the definition of $\not\subseteq$ gives

$$\exists x'. \left[\max(D'_{y_1}) - k_1 < x' \wedge y_1 - x' \geq k_1 \wedge y_2 \leq x' - k_2 \wedge y_2 < a_2\right] \vee$$

$$\exists x'. \left[\max(D'_{y_1}) - k_1 < x' \wedge y_1 - x' \geq k_1 \wedge y_2 \leq x' - k_2 \wedge x' - k_2 > b_2\right] \vee$$

$$\exists x'. \, [x' \leq b_2 + k_2 \wedge y_1 - x' \geq k_1 \wedge y_2 \leq x' - k_2 \wedge y_2 < a_2] \vee$$

$$\exists x'. \, [x' \leq b_2 + k_2 \wedge y_1 - x' \geq k_1 \wedge y_2 \leq x' - k_2 \wedge x' - k_2 > b_2] \ .$$

Now, $x' \leq b_2 + k_2$ conflicts $x' - k_2 > b_2$ in the last disjunct, which implies:

$$\exists x'. \left[\max(D'_{y_1}) - k_1 < x' \wedge y_1 - x' \geq k_1 \wedge y_2 \leq x' - k_2 \wedge y_2 < a_2\right] \vee$$

$$\exists x'. \left[\max(D'_{y_1}) - k_1 < x' \wedge y_1 - x' \geq k_1 \wedge y_2 \leq x' - k_2 \wedge x' - k_2 > b_2\right] \vee$$

$$\exists x'. \, [x' \leq b_2 + k_2 \wedge y_1 - x' \geq k_1 \wedge y_2 \leq x' - k_2 \wedge y_2 < a_2] \ .$$

We move x' and replace $<$ with $\leq$ as a preparation to eliminate x':

$$\exists x'.\, \left[\max(D'_{y_1}) - k_1 + 1 \leq x' \wedge \underline{y_1 - k_1 \geq x'} \wedge y_2 + k_2 \leq x' \wedge y_2 < a_2\right] \vee$$
$$\exists x'.\, \left[\max(D'_{y_1}) - k_1 + 1 \leq x' \wedge \underline{y_1 - k_1 \geq x'} \wedge y_2 + k_2 \leq x' \wedge x' \geq b_2 + k_2 + 1\right] \vee$$
$$\exists x'.\, \left[x' \leq b_2 + k_2 \wedge y_1 - k_1 \geq x' \wedge \underline{y_2 + k_2 \leq x'} \wedge y_2 < a_2\right] \ .$$

We eliminate x' in:

$$\left[\max(D'_{y_1}) - k_1 + 1 \leq y_1 - k_1 \wedge y_2 + k_2 \leq y_1 - k_1 \wedge y_2 < a_2\right] \vee$$
$$\left[\max(D'_{y_1}) - k_1 + 1 \leq y_1 - k_1 \wedge y_2 + k_2 \leq y_1 - k_1 \wedge y_1 - k_1 \geq b_2 + k_2 + 1\right] \vee$$
$$\left[y_2 + k_2 \leq b_2 + k_2 \wedge y_1 - k_1 \geq y_2 + k_2 \wedge y_2 < a_2\right] \ .$$

After some normalization:

$$\left[\max(D'_{y_1}) + 1 \leq y_1 \wedge y_2 + k_2 \leq y_1 - k_1 \wedge y_2 < a_2\right] \vee$$
$$\left[\max(D'_{y_1}) + 1 \leq y_1 \wedge y_2 + k_2 \leq y_1 - k_1 \wedge y_1 - k_1 \geq b_2 + k_2 + 1\right] \vee$$
$$\left[y_2 \leq b_2 \wedge y_1 - k_1 \geq y_2 + k_2 \wedge y_2 < a_2\right] \ .$$

In the last disjunct, since $y_2 < a_2$ and we know that $a_2 \leq b_2$ then $y_2 \leq b$ is redundant and the last disjunct becomes

$$y_1 - k_1 \geq y_2 + k_2 \wedge y_2 < a_2 \ .$$

This subsumes the first disjunct:

$$\max(D'_{y_1}) + 1 \leq y_1 \wedge y_2 + k_2 \leq y_1 - k_1 \wedge y_2 < a_2$$

So we are left with

$$\left[\max(D'_{y_1}) + 1 \leq y_1 \wedge \underline{y_2 + k_2 \leq y_1 - k_1} \wedge y_1 - k_1 \geq b_2 + k_2 + 1\right] \vee$$
$$\left[\underline{y_1 - k_1 \geq y_2 + k_2} \wedge y_2 < a_2\right] \ .$$

We now define $k_1 + k_2 = k^*$ which leaves us with:

$$\left[\max(D'_{y_1}) + 1 \leq y_1 \wedge \underline{y_2 + k^* \leq y_1} \wedge y_1 \geq b_2 + k^* + 1\right] \vee$$
$$\left[\underline{y_1 \geq y_2 + k^*} \wedge y_2 < a_2\right] \ .$$

Combining conjunctions:

$$\left[\underline{y_2 + k^* \leq y_1} \wedge y_1 \geq \max(b_2 + k^* + 1, \max(D'_{y_1}) + 1)\right] \vee$$
$$\left[\underline{y_1 \geq y_2 + k^*} \wedge y_2 < a_2\right] \ .$$

This can be written as a interval constraint:

$$[y_2 + k^*, y_1] \not\subseteq \left[a_2 + k^*, \max(b_2 + k^*, \max(D'_{y_1}))\right]$$

We can subtract k^* from all sides and get:

$$[y_2, y_1 - k_1 - k_2] \not\subseteq \left[a_2, \max(b_2, \max(D'_{y_1}) - k_1 - k_2)\right]$$

The rule then becomes

$$\frac{y_1 - x \geq k_1 \qquad [y_2, x - k_2] \not\subseteq [a_2, b_2]}{\begin{array}{l} b_2 + k_2 < x \leq \max(D'_{y_1}) - k_1 \vee \\ [y_2, y_1 - k_1 - k_2] \not\subseteq \left[a_2, \max(b_2, \max(D'_{y_1}) - k_1 - k_2)\right] \end{array}}$$

Rule R5: $c_1 \doteq [y_1, x] \not\subseteq [a_1, b_1] \qquad c_2 \doteq [x, y_2] \not\subseteq [a_2, b_2]$

We demand that at the point of conflict, replacing either one of the constraints with a plain $\leq$ will not detect a conflict. If replacing with $\leq$ detects the conflict then we simply employ R2. This means that $[y_1, x] \not\subseteq [a_1, b_1]$ removes at least one value from D'_x more than $y_1 \leq x$. The extra value removed γ should satisfy

$$\min(D'_{y_1}) \leq \gamma \wedge \forall y_1 \in D'_{y_1}.[y_1, \gamma] \subseteq [a_1, b_1] \ .$$

This means that

$$\min(D'_{y_1}) \leq \gamma \wedge [\min(D'_{y_1}), \gamma] \subseteq [a_1, b_1] \ .$$

From this we conclude that

$$a_1 \leq \min(D'_{y_1}) \leq \gamma \leq b_1 \ .$$

Similarly from the second constraint we conclude that

$$a_2 \leq \max(D'_{y_2}) \leq b_2 \ .$$

We require that neither constraints detect a conflict by themselves with D'_{y_1}, D'_{y_2}, and D'_x, but when propagated consecutively they reach a conflict. This means that there is at least one value $\gamma \in D'_x$ such that

$$\exists y_1 \in D'_{y_1}.[y_1, \gamma] \not\subseteq [a_1, b_1] \wedge$$

$$\forall y_2 \in D'_{y_2}.[\gamma, y_2] \subseteq [a_2, b_2]$$

This can be simplified to

$$[\min(D'_{y_1}), \gamma] \not\subseteq [a_1, b_1] \wedge [\gamma, \max(D'_{y_2})] \subseteq [a_2, b_2] \ .$$

If such γ exists then it can be equal to $\max(D'_x)$ such that

$$[\min(D'_{y_1}), \max(D'_x)] \not\subseteq [a_1, b_1] \wedge [\max(D'_x), \max(D'_{y_2})] \subseteq [a_2, b_2] \ .$$

To summarize, we have concluded that

$$a_1 \leq \min(D'_{y_1}) \leq b_1 \wedge [\min(D'_{y_1}), \max(D'_x)] \not\subseteq [a_1, b_1] \wedge$$

$$[\max(D'_x), \max(D'_{y_2})] \subseteq [a_2, b_2] \wedge a_2 \leq \max(D'_{y_2}) \leq b_2 \ .$$

The interval expressions can be expanded such that

$$a_1 \leq \min(D'_{y_1}) \leq b_1 \wedge \min(D'_{y_1}) \leq \max(D'_x) \wedge$$

$$(\min(D'_{y_1}) < a_1 \vee \max(D'_x) > b_1) \wedge (\max(D'_x) > \max(D'_{y_2})$$

$$\vee (\max(D'_x) \geq a_2 \wedge \max(D'_{y_2}) \leq b_2)) \wedge a_2 \leq \max(D'_{y_2}) \leq b_2 \ .$$

This is simplified to

$$a_1 \leq \min(D'_{y_1}) \leq b_1 < \max(D'_x) \wedge \max(D'_x) \geq a_2 \wedge a_2 \leq \max(D'_{y_2}) \leq b_2 \ .$$

We will depict this as

$$\psi \triangleq (a_1 \leq \min(D'_{y_1}) \leq b_1 < \max(D'_x) \wedge \max(D'_x) \geq a_2 \wedge a_2 \leq \max(D'_{y_2}) \leq b_2) \ . \quad (3.27)$$

We require that the resulting constraint be of the form

$$x \notin \mathcal{X} \vee [y_1, y_2] \not\subseteq [a^*, b^*] \quad ,$$

and meet the soundness and completeness requirements. First we look for $\mathcal{X}$which is defined by

$$\mathcal{X} = \left\{ x' | \forall y_1 \in D'_{y_1} \forall y_2 \in D'_{y_2}. \, [[y_1, x'] \subseteq [a_1, b_1] \vee [x', y_2] \subseteq [a_2, b_2]] \right\} \ .$$

We expand the interval operators to

$$\mathcal{X} = \{x' | \forall y_1 \in D'_{y_1} \forall y_2 \in D'_{y_2}.$$
$$y_1 > x' \vee (a_1 \leq y_1 \wedge x' \leq b_1) \vee x' > y_2 \vee (a_2 \leq x' \wedge y_2 \leq b_2)\} \ .$$

Since y_1 is bounded only from below and y_2 only from above then we can safely rewrite this expression for the worse cases of y_1 and y_2 which are $\min(D'_{y_1})$ and $\max(D'_{y_2})$:

$$\mathcal{X} = \{x' | \min(D'_{y_1}) > x' \vee (a_1 \leq \min(D'_{y_1}) \wedge x' \leq b_1) \vee$$
$$x' > \max(D'_{y_2}) \vee (a_2 \leq x' \wedge \max(D'_{y_2}) \leq b_2)\} \ .$$

From the initial assumptions, as expressed by Equation (3.27), we know that $a_1 \leq \min(D'_{y_1})$ and $\max(D'_{y_2}) \leq b_2$ are true. From, this we conclude that that $(a_1 \leq \min(D'_{y_1}) \wedge x' \leq b_1)$ can be simplified to $x' \leq b_1$ and that $(a_2 \leq x' \wedge \max(D'_{y_2}) \leq b_2)$ can be simplified to $a \leq x'$. This yields

$$\mathcal{X} = \left\{x' | \min(D'_{y_1}) > x' \vee x' \leq b_1 \vee x' > \max(D'_{y_2}) \vee a_2 \leq x'\right\} \ .$$

Equation (3.27) also states that $\min(D'_{y_1}) \leq b_1$ and $a_2 \leq \max(D'_{y_2})$. This makes both $\min(D'_{y_1}) > x'$ and $x' > \max(D'_{y_2})$ redundant in the expression above. So the expression becomes

$$\mathcal{X} = \{x' | x' \leq b_1 \vee a_2 \leq x'\} \ .$$

$$X = (-\infty, b_1] \cup [a_2, \infty) \ .$$

We will look for the values of a^* and b^* which produce the strongest $[y_1, y_2] \not\subseteq [a^*, b^*]$ part which meets our requirements. This can be achieved by finding the smallest a^* and the biggest possible b^*. First we require that the constraint is sound, i.e., derivable from the two original constraints:

$$\forall x \forall y_1 \forall y_2. \, [\psi \wedge ([y_1, x] \not\subseteq [a_1, b_1] \wedge [x, y_2] \not\subseteq [a_2, b_2]) \rightarrow (x \notin \mathcal{X} \vee [y_1, y_2] \not\subseteq [a^*, b^*])] \ .$$

Where ψ is defined in Equation (3.27). At first glance it seems that ψ does not effect the expression. We will omit ψ as it will, at most, strengthen the expression. Assume by negation that this is not so, i.e., there are x, y_1, y_2 that violate it. This means that

$$[y_1, x] \not\subseteq [a_1, b_1] \wedge [x, y_2] \not\subseteq [a_2, b_2] \wedge (x \leq b_1 \vee a_2 \leq x) \wedge [y_1, y_2] \subseteq [a^*, b^*] \ .$$

From $[y_1, x] \not\subseteq [a_1, b_1] \wedge [x, y_2] \not\subseteq [a_2, b_2]$ we conclude that $y_1 \leq x \leq y_2$. This means that $[y_1, y_2] \subseteq [a^*, b^*]$ can be replaced with $y_1 \geq a^* \wedge y_2 \leq b^*$:

$$[y_1, x] \not\subseteq [a_1, b_1] \wedge [x, y_2] \not\subseteq [a_2, b_2] \wedge (x \leq b_1 \vee a_2 \leq x) \wedge y_1 \geq a^* \wedge y_2 \leq b^* \ .$$

The best case scenario is when $y_1 = a^*$ and $y_2 = b^*$ as they maximize the intervals $[y_1, x]$ and $[x, y_2]$. This gives

$$[a^*, x] \not\subseteq [a_1, b_1] \wedge [x, b^*] \not\subseteq [a_2, b_2] \wedge (x \leq b_1 \vee a_2 \leq x) \ .$$

Consider the two possibilities for x, as expressed in the above expression in $x \leq b_1 \vee a_2 \leq x$. If $x \leq b_1$ then $[a^*, x] \not\subseteq [a_1, b_1]$ implies $a^* < a_1$ and if $x \geq a_2$ then $[x, b^*] \not\subseteq [a_2, b_2]$ implies $b^* > b_2$. In order to violate this, i.e., there will be no x to satisfy this equation, we require that $a^* \geq a_1 \wedge b_2 \leq b^*$. So it seems that the strongest constraint that matches the requirements is

$$x \notin \mathcal{X} \vee [y_1, y_2] \not\subseteq [a_1, b_2] \ .$$

When

$$\mathcal{X} = \{x' | x' \leq b_1 \vee a_2 \leq x'\} \ .$$

- Soundness: We need to check if

$$\begin{gathered}\big(a_1 \leq \min(D'_{y_1}) \leq b_1 \wedge a_2 \leq \max(D'_{y_2}) \leq b_2 \wedge [y_1, x] \not\subseteq [a_1, b_1] \wedge [x, y_2] \not\subseteq [a_2, b_2]\big) \\ \models \\ (x \notin \mathcal{X} \vee [y_1, y_2] \not\subseteq [a_1, b_2])\end{gathered}$$

  By negation assume that this is not so, i.e., there are x, y_1, y_2 such that

$$\begin{gathered}a_1 \leq \min(D'_{y_1}) \leq b_1 \wedge a_2 \leq \max(D'_{y_2}) \leq b_2 \wedge [y_1, x] \not\subseteq [a_1, b_1] \wedge [x, y_2] \not\subseteq [a_2, b_2] \quad \wedge \\ x \in \mathcal{X} \wedge [y_1, y_2] \subseteq [a_1, b_2] \quad .\end{gathered}$$

  We expand the definition of $\mathcal{X}$ and get

$$\begin{gathered}a_1 \leq \min(D'_{y_1}) \leq b_1 \wedge a_2 \leq \max(D'_{y_2}) \leq b_2 \wedge [y_1, x] \not\subseteq [a_1, b_1] \wedge [x, y_2] \not\subseteq [a_2, b_2] \quad \wedge \\ (b_1 \geq x \vee x \geq a_2) \wedge [y_1, y_2] \subseteq [a_1, b_2] \quad .\end{gathered}$$

  If $x \leq b_1$ then $[y_1, x] \not\subseteq [a_1, b_1]$ implies $y_1 < a_1$ which conflicts $[y_1, y_2] \subseteq [a_1, b_2]$. Otherwise, $a_2 \geq x$ and $[x, y_2] \not\subseteq [a_2, b_2]$ implies $y_2 > b_2$ which conflicts $[y_1, y_2] \subseteq [a_1, b_2]$. This means that there exists no x that satisfies the above expression, which conflicts our assumption that the new constraint is not derived from the original two interval constraints.

- Completeness: We need to check that

$$\psi \rightarrow \left(x \notin \mathcal{X} \vee [y_1, y_2] \not\subseteq [a_1, b_2], D'_x, D'_{y_1} D'_{y_2} \vdash \emptyset\right) \ .$$

  Where ψ is an expression defined by Equation (3.27). By negation we assume that

$$\psi \rightarrow \exists x \in D'_x \exists y_1 \in D'_{y_1} \exists y_2 \in D'_{y_2}. \left[x \notin \mathcal{X} \vee [y_1, y_2] \not\subseteq [a_1, b_2]\right] \ .$$

  The part of x can be simplified such that

$$\psi \rightarrow \exists y_1 \in D'_{y_1} \exists y_2 \in D'_{y_2}. \left[D'_x \setminus \mathcal{X} \neq \emptyset \vee [y_1, y_2] \not\subseteq [a_1, b_2]\right] \ .$$

  But according to the construction of $\mathcal{X}$ we know that $D'_x \setminus \mathcal{X} = \emptyset$, so the expression is further simplified to

$$\psi \rightarrow \exists y_1 \in D'_{y_1} \exists y_2 \in D'_{y_2}. [y_1, y_2] \not\subseteq [a_1, b_2] \ .$$

  We expand it according to the definition of interval constraints:

$$\psi \rightarrow \exists y_1 \in D'_{y_1} \exists y_2 \in D'_{y_2}. \left[y_1 \leq y_2 \wedge (y_1 < a_1 \vee b_2 < y_2)\right] \ .$$

  Since y_1 is bounded only from above and y_2 is bounded from below then we can replace y_1 with $\min(D'_{y_1})$ and y_2 with $\max(D'_{y_2})$:

$$\psi \rightarrow \left[\min(D'_{y_1}) \leq \max(D'_{y_2}) \wedge (\min(D'_{y_1}) < a_1 \vee b_2 < \max(D'_{y_2}))\right] \ .$$

  From the definition of ψ we know that $\min(D'_{y_1}) < a_1$ and $b_2 < \max(D'_{y_2})$ are false, so $(\min(D'_{y_1}) < a_1 \vee b_2 < \max(D'_{y_2}))$ is false and the expression above is falsified. This means that our assumption was wrong, i.e., our completeness requirement is not violated.

To summarize this part, the Combination rule is

$$\frac{[y_1, x] \not\subseteq [a_1, b_1] \qquad [x, y_2] \not\subseteq [a_2, b_2] \qquad \psi}{a_2 > x > b_1 \vee [y_1, y_2] \not\subseteq [a_1, b_2]} \ . \tag{3.28}$$

Rule R6: $c_1 \doteq [x, y] \not\subseteq [a_1, b_1] \qquad c_2 \doteq [y, x] \not\subseteq [a_2, b_2]$

We have $c_1 \wedge c_2 \rightarrow x = y$, because otherwise, e.g., if $x < y$, then c_2 is trivially false. Since $x = y$ then their joint value cannot be contained in either of the ranges $[a_1, b_1], [a_2, b_2]$. Hence,

$$c_{12} = (x = y) \wedge x \notin [a_1, b_1] \wedge x \notin [a_2, b_2] \ , \tag{3.29}$$

which implies that

$$\begin{aligned}\mathcal{X} &= \{x' \mid \forall y \in D'_y.\ x \neq y \vee x \in [a_1, b_1] \vee x \in [a_2, b_2]]\} \\ &= D'^C_y \cup [a_1, b_1] \cup [a_2, b_2]\ .\end{aligned}$$

The consequent of *Combine* is

$$\begin{aligned}c^* &= (x \notin \mathcal{X} \vee \exists x \in \mathcal{X}.\ x = y \wedge x \notin [a_1, b_1] \wedge x \notin [a_2, b_2]) \\ &= (x \notin \mathcal{X} \vee (y \in \mathcal{X} \wedge y \notin [a_1, b_1] \wedge y \notin [a_2, b_2])) \\ &= (x \notin \mathcal{X} \vee ((y \in [a_1, b_1] \vee y \in [a_2, b_2] \vee y \notin D'_y) \wedge y \notin [a_1, b_1] \wedge y \notin [a_2, b_2])) \\ &= (x \notin \mathcal{X} \vee (y \notin D'_y \wedge y \notin [a_1, b_1] \wedge y \notin [a_2, b_2])) \\ &= (x \in (D'_y \setminus ([a_1, b_1] \cup [a_2, b_2])) \vee y \notin (D'_y \cup [a_1, b_1] \cup [a_2, b_2]))\ .\end{aligned}$$

Hence, the resulting rule in this case is

$$\frac{[x, y] \not\sqsubseteq [a_1, b_1] \qquad [y, x] \not\sqsubseteq [a_2, b_2]}{(x \in (D'_y \setminus ([a_1, b_1] \cup [a_2, b_2])) \vee y \notin (D'_y \cup [a_1, b_1] \cup [a_2, b_2]))} \tag{3.30}$$

3.4.3 Selected rules not based on *Combine*

Rule R3: $c_1 \doteq y_1 \leq x \qquad c_2 \doteq [x, y_2] \not\sqsubseteq [a, b]$

We assume that at the point of conflict, replacing c_2 with $x \leq y_2$ makes c_{12} too weak to detect the conflict. Otherwise we simply use rule R2. Based on this assumption, which we denote by ψ, we now develop $\mathcal{X}$. ψ means that $[x, y_2] \not\sqsubseteq [a, b]$ removes at least one value from D'_x more than $x \leq y_2$. The extra value removed α should satisfy

$$\alpha \leq \max(D'_{y_2}) \wedge \forall y_2 \in D'_{y_2}.[\alpha, y_2] \subseteq [a, b]\ .$$

This means that

$$\alpha \leq \max(D'_{y_2}) \wedge [\alpha, \max(D'_{y_2})] \subseteq [a, b]\ .$$

From this we conclude that

$$a \leq \alpha \leq \max(D'_{y_2}) \leq b\ .$$

We require that neither constraints detect a conflict by themselves with D'_{y_1}, D'_{y_2}, and D'_x, but when propagated consecutively they reach a conflict. This means that there is at least one value $\alpha \in D'_x$ such that

$$\exists y_1 \in D'_{y_1}.y_1 \leq \alpha \wedge$$

$$\forall y_2 \in D'_{y_2}.[\alpha, y_2] \subseteq [a, b]$$

This can be simplified to

$$\min(D'_{y_1}) \leq \alpha \wedge [\alpha, \max(D'_{y_2})] \subseteq [a, b] \ .$$

If such α exists then it can be equal to $\max(D'_x)$ such that

$$\min(D'_{y_1}) \leq \max(D'_x) \wedge [\max(D'_x), \max(D'_{y_2})] \subseteq [a, b] \ .$$

To summarize, we have concluded that

$$a \leq \max(D'_{y_2}) \leq b \wedge \min(D'_{y_1}) \leq \max(D'_x) \wedge [\max(D'_x), \max(D'_{y_2})] \subseteq [a, b] \ .$$

The interval expression can be expanded such that

$$a \leq \max(D'_{y_2}) \leq b \wedge \min(D'_{y_1}) \leq \max(D'_x) \wedge (\max(D'_x) > \max(D'_{y_2}) \vee$$
$$(a \leq \max(D'_x) \wedge \max(D'_{y_2}) \leq b)) \ .$$

This is simplified to

$$a \leq \max(D'_{y_2}) \leq b \wedge \min(D'_{y_1}) \leq \max(D'_x) \wedge (\max(D'_x) > \max(D'_{y_2}) \vee a \leq \max(D'_x)) \ .$$

Since $a \leq \max(D'_{y_2})$ then if $\max(D'_x) > \max(D'_{y_2})$ is satisfied then $\max(D'_x) > \max(D'_{y_2}) \geq a$, i.e., $D'_x \geq a$, is implied. This means that $\max(D'_x) > \max(D'_{y_2})$ is redundant in the above expression, generating:

$$a \leq \max(D'_{y_2}) \leq b \wedge \min(D'_{y_1}) \leq \max(D'_x) \wedge a \leq \max(D'_x) \ .$$

We will depict this as

$$\psi' \triangleq (a \leq \max(D'_{y_2}) \leq b \wedge \min(D'_{y_1}) \leq \max(D'_x) \wedge a \leq \max(D'_x)) \ . \tag{3.31}$$

We require that the resulting constraint be of the form

$$x \notin \mathcal{X} \vee [y_1, y_2] \not\subseteq [a^*, b^*] \quad ,$$

and meet the soundness and completeness requirements. First we look for $\mathcal{X}$ which is defined by

$$\mathcal{X} = \left\{ x' | \forall y_1 \in D'_{y_1} \forall y_2 \in D'_{y_2} . \, [y_1 > x' \vee [x', y_2] \subseteq [a, b]] \right\} \ .$$

We expand the interval operator to

$$\mathcal{X} = \left\{ x' | \forall y_1 \in D'_{y_1} \forall y_2 \in D'_{y_2} . \, [y_1 > x' \vee x' > y_2 \vee (a \leq x' \wedge y_2 \leq b)] \right\} \ .$$

Since y_1 is bounded only from below and y_2 only from above then we can safely rewrite this expression for the worse cases of y_1 and y_2 which are $\min(D'_{y_1})$ and $\max(D'_{y_2})$:

$$\mathcal{X} = \left\{x' | \min(D'_{y_1}) > x' \vee x' > \max(D'_{y_2}) \vee (a \leq x' \wedge \max(D'_{y_2}) \leq b)\right\} .$$

From the initial assumptions, as expressed by Equation (3.31), we know that $\max(D'_{y_2}) \leq b$ is true. From, this we conclude that that $(a \leq x' \wedge \max(D'_{y_2}) \leq b)$ can be simplified to $a \leq x'$. This yields

$$\mathcal{X} = \left\{x' | \min(D'_{y_1}) > x' \vee x' > \max(D'_{y_2}) \vee a \leq x'\right\} .$$

Equation (3.31) also states that $a \leq \max(D'_{y_2})$ which makes $x' > \max(D'_{y_2})$ redundant in $x' > \max(D'_{y_2}) \vee a \leq x$. So the expression becomes

$$\mathcal{X} = \{x' \mid x' < \min(D'_{y_1}) \vee a \leq x'\} . \tag{3.32}$$

We propose the following consequent:

$$\begin{aligned} c^* &\doteq x \notin \mathcal{X} \vee [y_1, y_2] \not\subseteq [\min(D'_{y_1}), b] && (3.33)\\ &= a > x \geq \min(D'_{y_1}) \vee [y_1, y_2] \not\subseteq [\min(D'_{y_1}), b] . && (3.34) \end{aligned}$$

Note that c^* still follows our general pattern, by which the pivot is separated and not referred-to by the other disjunct. Since we cannot rely on the correctness of the general rule, we now prove that (3.34) satisfies (3.8) and (3.9):

- Eq. (3.8): Falsely assume the contrary, i.e., there are x, y_1, y_2 such that

  $$a \leq \max(D'_{y_2}) \leq b \wedge \min(D'_{y_1}) \leq \max(D'_x) \wedge a \leq \max(D'_x) \wedge y_1 \leq x$$
  $$\wedge [x, y_2] \not\subseteq [a, b] \wedge x \in \mathcal{X} \wedge [y_1, y_2] \subseteq [\min(D'_{y_1}), b] .$$

  Expanding $\mathcal{X}$ yields

  $$a \leq \max(D'_{y_2}) \leq b \wedge \min(D'_{y_1}) \leq \max(D'_x) \wedge a \leq \max(D'_x) \wedge y_1 \leq x$$
  $$\wedge [x, y_2] \not\subseteq [a, b] \wedge (x < \min(D'_{y_1}) \vee a \leq x) \wedge [y_1, y_2] \subseteq [\min(D'_{y_1}), b] .$$

  If $x < \min(D'_{y_1})$ then $y_1 \leq x$ implies $y_1 < \min(D'_{y_1})$ which conflicts $[y_1, y_2] \subseteq [\min(D'_{y_1}), b]$. Otherwise, $x \geq a$ and $[x, y_2] \not\subseteq [a, b]$ implies $y_2 > b$ which conflicts $[y_1, y_2] \subseteq [\min(D'_{y_1}), b]$.

- Eq. (3.9): We need to check that

  $$\psi' \rightarrow (x \notin \mathcal{X} \vee [y_1, y_2] \not\subseteq [\min(D'_{y_1}), b], D'_x, D'_{y_1} D'_{y_2} \vdash \emptyset) ,$$

where ψ' is an expression defined by Equation (3.31). Falsely assume that

$$\psi' \to \exists x \in D'_x \exists y_1 \in D'_{y_1} \exists y_2 \in D'_{y_2}.\ x \notin \mathcal{X} \vee [y_1, y_2] \not\subseteq [\min(D'_{y_1}), b]\ .$$

The part of x can be simplified such that

$$\psi' \to \exists y_1 \in D'_{y_1} \exists y_2 \in D'_{y_2}.\ D'_x \setminus \mathcal{X} \neq \emptyset \vee [y_1, y_2] \not\subseteq [\min(D'_{y_1}), b]\ .$$

But according to the construction of $\mathcal{X}$ we know that $D'_x \setminus \mathcal{X} = \emptyset$, so the expression is further simplified to

$$\psi' \to \exists y_1 \in D'_{y_1} \exists y_2 \in D'_{y_2}.\ [y_1, y_2] \not\subseteq [\min(D'_{y_1}), b]\ ,$$

or

$$\psi' \to \exists y_1 \in D'_{y_1} \exists y_2 \in D'_{y_2}.\ y_1 \leq y_2 \wedge (y_1 < \min(D'_{y_1}) \vee b < y_2)\ .$$

Since y_1 is bounded only from above and y_2 is bounded from below then we can replace y_1 with $\min(D'_{y_1})$ and y_2 with $\max(D'_{y_2})$:

$$\psi' \to [\min(D'_{y_1}) \leq \max(D'_{y_2}) \wedge (\min(D'_{y_1}) < \min(D'_{y_1}) \vee b < \max(D'_{y_2}))]\ .$$

$\min(D'_{y_1}) < \min(D'_{y_1})$ is clearly false, and from the definition of ψ' we know that $b < \max(D'_{y_2})$ is false. This falsifies $\min(D'_{y_1}) < \min(D'_{y_1}) \vee b < \max(D'_{y_2})$ from the expression, hence the whole expression is also falsified. This means that our assumption was wrong, i.e., (3.9) is not violated.

To summarize, the rule is

$$\frac{y_1 \leq x \qquad [x, y_2] \not\subseteq [a, b] \qquad \psi}{a > x \geq \min(D'_{y_1}) \vee [y_1, y_2] \not\subseteq [\min(D'_{y_1}), b]}\ . \tag{3.35}$$

Rule R7: $c_1 \doteq (y \leq x + k_1) \qquad c_2 = (x \leq y + k_2)$

Isolating $x - y$ on both sides yields $c_{12}(x, y) = -k_1 \leq x - y \leq k_2$, which is false if $k_1 + k_2 < 0$. Since it is simply a conjunction of the input constraints, then (3.8) and (3.9) are satisfied trivially.

3.5 Experimental results

We compared three different settings: (1) HCSP with general constraints learning based on Combine (from hereon—HCSP), (2) HCSP using only clause-based learning with

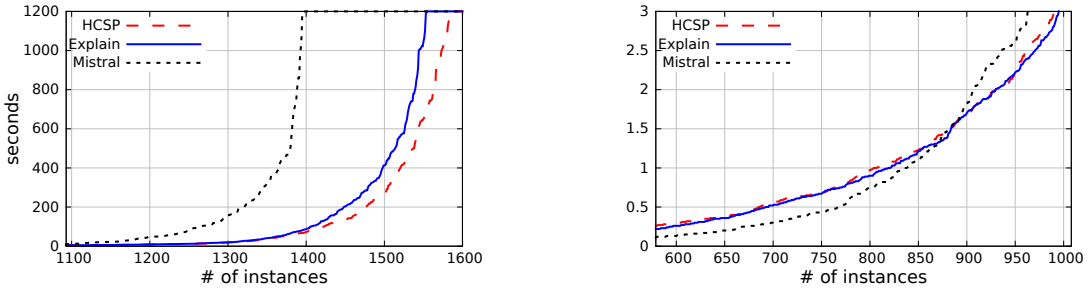


Figure 3.2: Number of instances solved within the given time limit comparing HCSP, EXPLAIN, and MISTRAL (a) Shows the time in linear scale; (b) A zoom-in of the left figure showing the cross-over between MISTRAL and HCSP occurring after 1-2 seconds.

explanations, as described in Sec. 3.2 (from hereon—EXPLAIN)[4] (3) MISTRAL [Heb08] latest version (1.550).

To evaluate the three alternatives we used a subset of benchmarks of the Fourth International CSP Solver competition [C0909]. Specifically out of over 7000 in the competition's satisfiability benchmark-set, we focused on the 2162 benchmarks that have at least one comparison operator from $\{<, \leq, \geq, >\}$ (the reason being that the rules in Table 3.1 refer to combinations of constraints based on these operators and constraints that are consequents of these rules).

All tools were compiled with gcc-4.8.1 for 32-bit Intel architecture, and were run on a 4 core Intel® Xeon® 2.5GHz with 3GiB RAM. We have set a hard limit of 1GiB memory usage and 1200 seconds of CPU-time. Out of memory and time-outs are called 'fails' in the discussion below.

Fig. 3.2 compares the three engines. Number of fails in HCSP was 25% less than MISTRAL. Number of fails when using Combine was 4.9% less than EXPLAIN. In Fig.3.3(a) we see that the conflict analysis of HCSP is not beneficial for runs below 1.5 seconds, where MISTRAL ran faster.

Fig. 3.3 compares the number of backtracks considering only non-failing runs in all solvers (log-scale). The average number of backtracks in HCSP is 2045, in EXPLAIN 4389, and in MISTRAL 49562. As noted in the introduction, this drastic difference in the average backtrack-count indicates that the cost of learning is compensated-for by a better search.

HCSP is written in C++, contains 23k lines of non-comment code, and its architecture

[4]We emphasize that this is a far-improved engine in comparison to four years ago [VS10a] owing to numerous optimizations that are beyond the scope of the current article, including an improved decision heuristic, better choice of explanations, and common-sub-expression elimination.

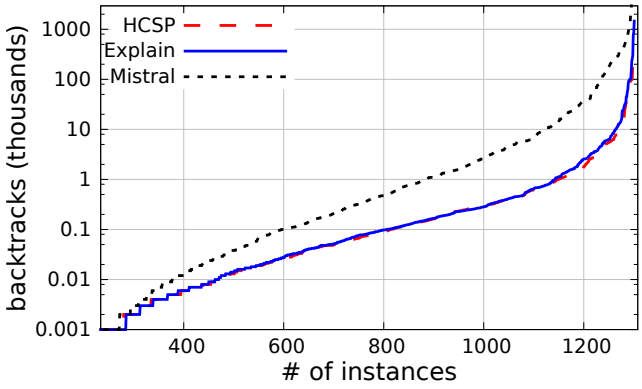


Figure 3.3: Comparing the number of backtracks for successful runs (log-scale).

enables the addition of new constraints and new rules without changing the core solver. It is free software [Vek] under the GPL license.

Conclusion and future work

We have presented a new learning scheme based on inference of general constraints. We presented the development of various inference rules that are necessary for this scheme, but it is clear that there is still a lot of work in deriving such rules for additional popular pairs of constraints which are currently not supported and force HCSP into a fallback solution. In addition, currently learning general constraints is incompatible with producing machine-checkable proofs in case the formula is unsatisfiable, in contrast to our earlier explanation-based method [VS10a].

Chapter 4

Architecture and Capabilities of HCSP

4.1 The solving algorithm of HCSP

4.2 Propagating constraints

AC3 [Mac77] maintains a queue of constraints which contains constraints that need to be revised. Every iteration of the algorithm removes a constraint from the queue and performs a *revise* operation over it, i.e., the algorithm propagates the constraint. Unlike AC3, HCSP maintains two queues:

- A queue of variables that have not been checked since their domains have been reduced. The algorithm removes a variable from the variable queue and revises the constraints affecting this variable. As an optimization, not all affecting constraints are revised, only those that meet the watch criterion. Watches are described in Subsection 4.3.4. This is the prime queue of the algorithm, i.e., in most cases this will be the only queue used during propagation.

- A queue of constraints that should be activated. The constraints in this queue are considered only when the variable queue is empty. A constraint may enter this queue only after it was encountered by the variable queue, and only if the constraint's propagation algorithm requests to do so.

This mechanism resembles Minion which, as described in [GMN], also uses a variable queue for all constraints except for `AllDifferent`which goes into the constraint queue. Many papers including [SS08] suggest using a priority constraint queue, where slow constraints get a lower priority. It would be interesting to attempt constraint priority queue for propagating constraints in HCSP.

The propagation algorithm in HCSP is described in Algorithm 4.1.

Algorithm 4.1 The HCSP constraint propagation algorithm

```
1: function PROPAGATE(C, V, D, Q_v)
2:     Q_c ← ∅
3:     while Q_v ≠ ∅ ∨ Q_c ≠ ∅ do
4:         if Q_v ≠ ∅ then
5:             (v, c) ← pop_front(Q_v)          ▷ If c ≠ ⊥, then c ∈ C inserted v into Q_v
6:             for all r ∈ Watched(v)\{c} do
7:                 result←REVISE (r, D, Q_v, Q_c)          ▷ propagate r and update Q_v, Q_c
8:                 if Empty(result) then
9:                     return CONFLICT
10:                end if
11:            end for
12:        else
13:            r ← pop_front(Q_c)
14:            result←REVISE (r, D, Q_v, Q_c)
15:            if Empty(result) then
16:                return CONFLICT
17:            end if
18:        end if
19:    end while
20: end function
21: function REVISE(r, D, Q_v, Q_c)
22:    result←propagate(r, D)
23:    if result = CONFLICT then
24:        return result
25:    else if result = POSTPONE then
26:        if r ∉ Q_c then Q_c.push_back(r)
27:        end if
28:    else
29:        D ← D apply changes from result.
30:        for all v' modified by result do
31:            push_back(Q_v,(v', r))
32:        end for
33:    end if
34: end function
```

4.2.1 The decision heuristic of HCSP

The decision heuristic of HCSP is based on a mix of Chaff's VSIDS [MMZ+01a], *dom/wdeg* strategy [BHLS04], and *phase saving* [Sht00] as follows. HCSP maintains a weight value for each variable and makes decisions based on that value. The weights, which are at the core of the decision heuristic, are floating-point values which are modified according to the following strategy:

- The initial weight of a variable is set to the number of constraints that constrain it.

- Each constraint that participates in conflict analysis increases the weights of all affected variables by a factor.

- After each backtrack the increase factor is increased by approximately 4%, effectively lowering the weights by 4%.

For a variable v that participated in a recent conflict analysis HCSP calculates $h(v) = |D_v|/\text{weight}(v)$, and for all other variables it calculates $h(v) = 8 * |D_v|/\text{weight}(v)$. Then the decision heuristics selects the variable with the lowest value of $h(v)$. Since a variable that participated in the latest conflict analysis gets lower $h(v)$ it has a bigger chance of being selected. Measurements show that this difference in $h(v)$ improves performance.

This is similar to the conclusions of [BS10] which showed that sometimes it is beneficent to retry the same variable after backtrack, and in other cases it is beneficent to try a different variable. They showed that instead of a-priory choosing either one of the methods it is better to combine both. They lean towards, without restricting to, repeating the same variable. Their experiments show that not only the combined method never performs much worse than the alternatives, it sometimes performs better than the alternatives.

The variable ordering policy of HCSP resembles *dom/wdeg* strategy of CSP, where both weight(v) and *wdeg* increase with conflicts. The two main differences are that: (1) *wdeg* counts only the conflicting constraints when weight(v) counts all constraint participating in conflict analysis; (2) weight(v), unlike *wdeg*, does not depend on the current domains of neighboring variables.

The decision is performed by Algorithm 4.2.

Algorithm 4.2 Value ordering heuristic in HCSP

```
1: function SelectValue(D_v)
2:     if previously-assigned then
3:         d ← previous(v)
4:         if d ∈ D_v then
5:             return d
6:         else if d ≥ min(D_v) ∧ d ≤ (min(D_v) * 2 + max(D_v))/3 then
7:             return min(D_v)                    ▷ The first 1/3 of the domain
8:         else if d ≤ max(D_v) ∧ d ≥ (max(D_v) * 2 + min(D_v))/3 then
9:             return max(D_v)                    ▷ The third 1/3 of the domain
10:        else if random(2) = 0 then
11:            return max(D_v)
12:        end if
13:    end if
14:    return min(D_v)
15: end function
```

Once the variable, v, is decided HCSP has to choose which value to assign to it. If the variable was previously assigned to value d and $d \in D_v$ then the decision is set to $v \leftarrow d$. Otherwise, the range of values $[\min(D_v), \max(D_v)]$ is partitioned into three partitions. If d is in the lower partition then the decision is $v \leftarrow \min(D_v)$, and if d is in the higher partition then the decision is $v \leftarrow \max(D_v)$, otherwise the value is selected randomly between $\min(D_v)$ and $\max(D_v)$. Lastly, if the variable was never assigned before then HCSP chooses the smallest value, i.e., $\min(D_v)$.

Experimentation with other heuristics showed that there is an advantage in selecting $\min(D_v)$ or $\max(D_v)$ over other values. For example, if $D_v = [1, 1000]$ and the decision $v \leftarrow 500$ leads to a conflict then conflict analysis may conclude that $v \neq 500$. This conclusion will create a hole in the domain such that $D_v = [1, 499] \cup [501, 1000]$. Decision that assign v values which are not on the boundaries of D_v create more holes in D_v. Domains with holes take more memory, possibly exhausting memory, and take more time to propagate. The added costs do not seem to be worth-while even in cases when the number of backtracks is lowered.

4.3 HCSP architecture

4.3.1 HCSP Domains

Holding domains efficiently is not trivial. Using C++'s `std::set<int>` will work reasonably well for small domains, but will be impractical for domains with hundreds of elements or more. There are many known alternatives for representing domains which are out of scope of this document. Instead, I will outline the way HCSP represents domains.

In HCSP there are two alternative representations for domains, `hcsp::big_iset` and `hcsp::small_iset`. Currently, it is decided at compilation time with which domain type to build HCSP. The representations are:

- `small_iset`represents sets restricted to the range $[0, 31]$. The set is implemented as 32 bit values, where bit $b_i = 1$ means that value i is in the set.

- `big_iset`is restricted to the range $[-2^{31}, 2^{31})$. The set is implemented as a collection of non-overlapping intervals. For example, $D = [1, 6] \cup [10, 100] \cup [102, 110]$. Because this representation is inefficient for the common case of strides, such as $1, 3, 5, \ldots, 101$, strides have a special representations. Every `big_iset`is restricted to one stride, e.g., $1, 3, 5, \ldots, 101$ is represented as $[1, 101]$ with skip size=2. For convenience, HCSP uses the Matlab syntax 1:2:101 to indicate the set $\{1, 3, 5, \ldots, 101\}$.

Currently, there is no compact representation for sets such as 1:3:101 ∪ 102:105:120 since it involves two different strides. This set is represented as an explicit list of 58 elements rather than two stride ranges. It will be a good project to rewrite `big_iset`as a specialized implementation of a balanced search tree rather than the current reliance on `std::map` which is slow and inflexible.

For the future it is a good idea to either get rid of `small_iset`or to make it work in combination with `big_iset`.

4.3.2 HCSP Constraints

Constraints are the most extensible interface of HCSP. New constraints can be written by extending the `hcsp::constraint_propagator` base-class without touching any other part of HCSP. The interface of `constraint_propagator` has many aspects and parts, most of which have reasonable defaults. There are only four methods that must be defined for new constraints:

- `propagate(constraint_runtime_info &info)`:
  This method defines how the constraint should be propagated. The propagator uses the `info` object to access and update variable domains. The only way propagator can access CSP variables is through `info`. The propagator does not know if `info` refers to CSP variables or to, e.g., mock variables used for testing the propagator.

  Note that this method may maintain internal data structure and modify it during propagation. For example, it may maintain two-watch literals [MMZ+01b] as in SAT.

- `conflicts(const constraint_runtime_info &info) const`:
  This method detects if `propagate` would detect a conflict on the current domains, and does this without modifying neither the domains nor the internal state of the propagator.

- `constraint_check(const constraint_runtime_info &info) const`:
  Assuming that `info` (there is a short description of `info` in `propagate()` above) contains fully assigned domains, tell if this assignment satisfies the constraint. This method is used mostly for the automated testing of `propagate()`.

- `clone() const`:
  Create an identical copy of the current propagator.

There are many other methods with reasonable defaults, which may be overridden. There are two reasons to override the default methods: (1) to overcome the poorer

complexity of the defaults; (2) to implement features such as `combine()` for conflict analysis.

4.3.3 Decomposing the CSP into basic HCSP constraints

HCSP decomposes complex constraints into basic constraints through reification [COC97], a process which is similar to Tseitin transformation in SAT. A reified constraint has the form

$$c(\bar{y}) \leftrightarrow x = 1 \quad .$$

Where $\bar{y}$ is a set of variables which x is not one of them, and $c(\bar{y})$ is a constraint over these variables.

In most cases HCSP prefers one-sided reification, akin to one-sided Tseitin, which has proven to make HCSP work faster. This transformation has the form $l \vee c(\bar{y})$, where l is a literal such as $x = 0$ or $x = 1$. The transformation decomposes the disjunction of two constraints $(x_1 \geq x_2 \vee x_3 \leq x_4 + x_5)$ into three individual constraints:

$$\begin{aligned} & D_{y_1}, D_{y_2} \in \{0, 1\} \\ & c_1 \doteq (y_1 = 0 \vee x_1 \geq x_2) \\ & c_2 \doteq (y_2 = 0 \vee x_3 \leq x_4 + x_5) \\ & c_3 \doteq (y_1 = 1 \vee y_2 = 1) \qquad . \end{aligned}$$

Where D_{y_1} and D_{y_2} are the domains of the auxiliary variables y_1 and y_2.

Besides reification, HCSP decomposes big arithmetic expressions into their basic operations. Consider for example:

$$x_1 \leq \text{abs}(x_2) + \max(x_3, x_4) \quad .$$

This constraint is broken up into three basic constraint while introducing auxiliary variables y_1, y_2, and y_3:

$$\begin{aligned} & y_1 \leq \text{abs}(x_2) && \wedge \\ & y_2 \leq \max(x_3, x_4) && \wedge \\ & y_3 \leq y_1 + y_2 && \wedge \\ & y_3 - x_1 \geq 0 && . \end{aligned}$$

This decomposition is slightly wasteful since it could have used x_3 instead of introducing y_3. This waste does not seem to impact HCSP performance due to the solving process, and it simplifies common sub-expression elimination.

This is unlike the standard conversion from `MiniZinc` to `FlatZinc`, which is to create constraints with '=' over '$\leq$'. Also, the `FlatZinc` specification[1] has, among others, only the equivalent of $\max(a, b) = c$ and does not have $\max(a, b) \geq c$. For the example above, the `FlatZinc` model will look like:

$$\begin{aligned} z_1 &= \text{abs}(x_2) \quad \wedge \\ z_2 &= \max(x_3, x_4) \quad \wedge \\ z_3 &= z_1 + z_2 \quad \wedge \\ x_1 &\leq z_3 \quad . \end{aligned}$$

The implication of the difference between `FlatZinc` and HCSP in this regard is that HCSP decomposition seemingly enlarges the search space since the auxiliary variables y_i are less constrained. It turns out that tighter decomposition hurts performance more than it helps due to: (1) processing '$\leq$' is much faster than '=', and (2) the way HCSP analyzes conflicts. In effect, conflict analysis can get rid of many auxiliary variables constrained by '$\leq$' , such as y_1, which are usually used as pivots in Combine or Resolve rules. On the other hand, with '=' it is more difficult for the conflict analysis to eliminate the more constrained auxiliary variables z_i .

4.3.4 Constraint watches

Constraint watches have a long history in SAT and CSP. The Chaff [MMZ+01b] SAT solver used a 2-watched literals scheme to make a major advancement in SAT solving. Similarly, a variety of watches has been defined in [SS04] and later used in Minion [GJM06] which had a significant performance improvement for constraint propagation.

A watch is an algorithmic and data-structure entity that connects a variable and a constraint and waits for a specific event. Consider for example the constraint $x \neq y$. If the domains are $D_x = \{1, 2, \ldots, 5\}$, $D_y = \{1, 2, \ldots, 5\}$ then $x \neq y$ is arc-consistent. It will remain arc-consistent for $D'_x = \{1, 2, 3, 4\}$, $D''_x = \{2, 3, 4\}$, or even $D'''_x = \{3, 4\}$. This constraint may stop being consistent only when $|D_x| = 1$ or $|D_y| = 1$ and not before. It makes no sense to visit $x \neq y$ before one of the variables is fully assigned. A `FINALIZATION_WATCH` watch over x will visit $x \neq y$ only when x is fully assigned, and similarly for a watch over y.

This is not a big win for a single constraint like $x \neq y$, but if there are many $\neq$ constraints connected to x then none of them would be revised before x is fully assigned.

[1] This example uses a more compact syntax than the verbose `FlatZinc` syntax, but the semantics is similar.

For that, HCSP groups similar watches together so that an event is tested only once. This way if there are n constraints $x \neq y_1$, $x \neq y_2$, ..., $x \neq y_n$, then it costs only $O(1)$ to maintain them when $|D_x| > 1$.

HCSP supports watches similar to Minion:

- `FINALIZATION_WATCH`, as discussed in the example, visits the constraint when the variable is fully assigned. (e.g. in $x \neq y$)
- `LOWER_BOUND_WATCH` visits the constraint when domain reduction changed the lower bound of the domain. (e.g. for x in $x \leq y$ after the initial propagation only changes to D_x that increase $\min(D_x)$ may cause $x \leq y$ to reduce D_y).
- `UPPER_BOUND_WATCH` visits the constraint when domain reduction changed the upper bound of the domain. This is the opposite of `LOWER_BOUND_WATCH`. (e.g. for y in $x \leq y$)
- `MODIFICATION_WATCH` visits the constraint every time the domain of the variable is reduced. This is the equivalent of the default behavior for solvers that do not support watches.
- `VALUE_WATCH`(t), for variable v, visits the constraint when value t is removed from domain D_v. A Boolean clause in HCSP would implement 2-watch literals by setting two value watches on two of its variables, waiting for either 0 or 1 to be removed from the {0,1} domain.

4.4 Solving optimization problems

4.4.1 Introduction to optimization

HCSP supports both minimization and maximization problems, but for simplicity we will describe only minimization. The solving algorithm for maximization is not much different.

A minimization problem has, on top of the constraints, a requirement to find a solution that minimizes an objective function $\mathcal{O}(v_1, \ldots, v_m)$. HCSP effectively minimizes the solution by minimizing the objective variable o that contains the evaluation of the objective variable $o = \mathcal{O}(v_1, \ldots, v_m)$.

Minimization (maximization) is done in two circles:

- The inner circle is the normal solving process with a specialized decision heuristic.
- The top-level circle reruns the solver with a smaller bound on the objective function.

The solver selects the decision variable using the same heuristic for minimization as for satisfiability. However, when it chooses the value to assign, it selects the smallest domain value instead of the best performing heuristic. By doing so HCSP strives to return a solution with a small value for the objective without much effort, and without any optimality claims.

Algorithm 4.3 Optimizing an objective variable o in HCSP

```
 1: function OPTIMIZE(C, V, D, o)
 2:     best←UNSAT
 3:     while true do
 4:         result ← solve(C, V, D, o)
 5:         if result ∈ {UNSAT, TIMEOUT} ∧ best =UNSAT then
 6:             return ⟨result,{}⟩
 7:         end if
 8:         if result =UNSAT then
 9:             return ⟨OPTIMAL,best⟩
10:         end if
11:         if result = TIMEOUT then
12:             return ⟨SUBOPTIMAL,best⟩
13:         end if
14:         value ← result[o]                          ▷ o is part of the result
15:         best ← result
16:         if minimizing then
17:             D_o ← D_o ∩ (−∞, value − 1]            ▷ D was not affected by 'solve'
18:         else
19:             D_o ← D_o ∩ [value + 1, ∞)
20:         end if
21:         if D_o = ∅ then
22:             return ⟨OPTIMAL,best⟩
23:         end if
24:     end while
25: end function
```

The top level algorithm shown in Algorithm 4.3, optimizes o by repeatedly solving the CSP with smaller domains of o until the problem is no longer satisfiable. The last solution, before D_o is too small to be satisfiable, is the optimal one. Because HCSP only reduces the domain of o and does not assign a single value, some iterations will produce an assignment which is significantly better than the previous assignment. This, combined with the decision heuristic, lets HCSP make big steps towards the optimal value.

Successive iterations learn from the previous iterations:

- Conflict constraints are retained between iterations, helping to avoid repeating similar conflicts.

- Previous assignments to variables, i.e., *phase saving* [Sht00], such that the next iteration reaches a similar solution relatively fast.
- Counters used for decision heuristic are retained, so that previous conflicts affect decision making.

Experiments have shown that it may be beneficial to tune the decision heuristic for variables other than the optimization variable o. If the optimization function $\mathcal{O}(v_1, \ldots, v_m)$ is ascending[2] with respect to $v_1, \ldots, v_m$ then it is sometimes beneficial to prefer the smallest values of $v_1, \ldots, v_m$. Although promising, this direction has not been pursued and should be investigated in the future.

4.5 Generating an interpolant

McMillan [McM03] introduced an interpolating theorem prover. His technique made it possible to prove many types of formulas by using a SAT solver. It may be beneficial to extend this technique such that CSP would be used for instead of SAT. Consider an unsatisfiable Boolean CNF which can be split into two conjuncts $\psi \wedge \varphi$. According to [Cra57] it is possible to find a formula, an interpolant, I such that:

- $vars(I) \subseteq vars(\psi) \cap vars(\varphi)$
- $\psi \vdash I$
- $I \wedge \varphi \vdash \bot$

Algorithm 4.4, which is implemented in HCSP, can generate an interpolant for CSP using a technique similar to McMillan's. It is adopted from [McM05] where it is presented for Boolean-SAT, to the world of Signed-CNF SAT, without modifications. The algorithm and all the proofs in [McM05] are taken as is, where instead of Boolean-literals it uses signed-literals.

Algorithm 4.4 accepts G for the unsatisfiability of the CSP. Currently, the algorithm can work only on explanation-based proofs[3]. The proof G is a Directed Acyclic Graph (DAG) whose roots are constraints in $\mathcal{C}$ and the leaf node is $\bot$. For an explanation-based proof, if a root c_1 is a non-clausal constraint then every one of its outgoing edges is connected to singed-clause explanation, e.g., c_2. Every such explanation has only one incoming edge. Other than that, every inner node has two incoming edges which represent the binary-resolution that derives this node.

[2]The notion of ascending objective function can be extended to monotone functions, but it was left out because it complicates the description

[3]Future work: it would be interesting to lift this restriction and work with the more general Combine bases proofs.

Algorithm 4.4 The generation of an interpolant in HCSP

```
1: function INTERPOLATE(ψ, φ, G)
2:     ψ' ← ψ, φ' ← φ, and G' ← G.
3:     for all r ∈ G s.t. r is a non-clausal constraint do
4:         X ← {e_i|(r, e_i) ∈ G'}                ▷ X gets the set of clauses adjacent to r
5:         Ω ← ⋀_{e_i∈X} e_i
6:         ψ' ← ψ' with r replaced by Ω.
7:         φ' ← φ' with r replaced by Ω.
8:         G' ← G' \ {r} \ {(r, e_i)|(r, e_i) ∈ G'}
9:     end for
10:    return INTERPOLATE-SIGNED-CLAUSE (ψ', φ', G')
11: end function
```

Theorem 4.5.1 (INTERPOLATE is correct) *We assume that ϕ and φ are conjunctions of non-overlapping sets of constraints.*

Given G, an explanation-based proof of $\psi \wedge \varphi \models \bot$, Algorithm 4.4 produces a valid interpolant for sub-problems ψ and φ.

Lemma 4.5.1 *Consider G, the explanation-based proof for $\psi \wedge \varphi \models \bot$, a non-clausal constraint in G depicted as r, and $(r, e_1), (r, e_2), \ldots, (r, e_n)$ the outgoing edges of r in G. Define G' as a sub-graph of G resulting from the removal of r and its outgoing edges. Define ψ', φ' as rewrites of ψ, φ where the occurrences[4] of r, if any, are replaced with $e_1 \wedge e_2 \wedge \cdots \wedge e_n$. Then $\psi' \wedge \varphi' \models \bot$ and G' is a proof for this expression.*

Proof To show that G' is a valid proof for $\psi' \wedge \varphi' \models \bot$ we traverse the proof from $\bot$ backwards. With G this traversal would stop at a set of roots $r_1, r_2, \ldots, r_k$ which are all variables of $\psi \wedge \varphi$. Without loss of generality, assume that the root removed from G' was $r_1 = r$. Traversing back on G' from $\bot$ would then give $r_2, \ldots, r_k$ and a subset of $e_1, e_2, \ldots, e_n$ instead of r_1, because $e_1, \ldots, e_n$ are the only successors to r_1.

Because $r_2, \ldots, r_k$ are part of $\psi \wedge \varphi$ and were not removed from $\psi' \wedge \varphi'$ then they are also part of $\psi' \wedge \varphi'$. Since $\psi' \wedge \varphi'$ is created by replacing r_1 with $e_1 \wedge e_2 \wedge \cdots \wedge e_n$ then $e_1, e_2, \ldots, e_n$ are also in $\psi' \wedge \varphi'$.

Since traversing the application of rules backwards from $\bot$ to the roots stops at variables of $\psi' \wedge \varphi'$ then it means that applying the rules on the variables of $\psi' \wedge \varphi'$ will reach the leaf node $\bot$. This shows that G' is a valid proof for $\psi' \wedge \varphi' \models \bot$. □

Proof [Proof of Theorem 4.5.1] By induction on the number of non-clausal constraints in G. Clearly if all constraints are clauses then this algorithm works correctly. Assuming

[4]Since ψ, φ are a partitioning of G, the constraint r is either in ψ or in φ.

that it works correctly for n non-clausal constraints in G we need to show that it works correctly also for $n+1$ non-clausal constraints.

Consider G with $n+1$ non-clausal constraints. The first iteration of the loop takes non-clausal constraint r, and removes it from G' and replaces it with $e_1 \wedge e_2 \wedge \cdots \wedge e_k$ in $\psi \wedge \varphi$. According to Lemma 4.5.1 we know that the resulting G', ψ', and φ' maintain the invariant that G' is a proof for $\psi' \wedge \varphi' \models \bot$.

According to the assumption that the algorithm works for n non-clausal constraints in G we know that it will produce a valid interpolant I for ψ' and φ' as set by the first iteration. This means that $\psi' \models I$ and $I \wedge \varphi' \models \bot$. We are left to show that I is also an interpolant for the original G, ψ, φ.

If ψ had r then it can be written as $\psi = (\bar{\psi} \wedge r)$ where $\bar{\psi}$ represents ψ with r removed. In this case $\psi' = (\bar{\psi} \wedge \Omega)$. To check the relation between ψ and ψ', look at the relation between r and Ω. Since all e_i are explanations to r we know that $\forall e_i \in X.[r \models e_i]$, i.e., $r \models \Omega$. This means that $(\bar{\psi} \wedge r) \models (\bar{\psi} \wedge \Omega)$, i.e., $\psi \models \psi'$ when r is in ψ. If r is not in ψ then $\psi' = \psi$ and trivially $\psi \models \psi'$. It can be shown, similarly, that $\varphi \models \varphi'$.

The following shows that I satisfies all three interpolation requirements for ψ and φ. Since $\psi \models \psi'$ and, according to the induction assumption, $\psi' \models I$ then $\psi \models I$. Since $\varphi \models \varphi'$ then $\varphi \wedge I \models \varphi' \wedge I$ and since, according to the induction assumption, $\varphi' \wedge I \models \bot$ it turns out that $\varphi \wedge I \models \bot$.

According to the induction assumption we know that I refers to variables shared by ψ' and φ'. If $\psi' = \bar{\psi} \wedge \Omega$ then $vars(\psi') = vars(\bar{\psi}) \cup vars(e_1) \cup \cdots \cup vars(e_n)$. Since e_i are explanations to r then $vars(e_i) \subseteq vars(r)$ and hence $vars(\psi') \subseteq vars(\bar{\psi}) \cup vars(r)$. And since, according to the induction assumption, $vars(I) \subseteq var(\psi')$ then $vars(I) \subseteq vars(\bar{\psi}) \cup vars(r)$. According to the definition of $\bar{\psi}$ this simplifies to $vars(I) \subseteq vars(\psi)$. Using the same technique it is easy to show that $vars(I) \subseteq vars(\varphi)$.

This means that I meets all three requirements for interpolants over $n+1$ non-clausal constraints in G. This proves by induction that the algorithm generates the correct I for any number of non-clausal constraints. $\square$

Appendix

Chapter A

Constraints which HCSP supports

[1] *Constraint propagation* and the difference between precise and imprecise propagation is defined in Sect. 1.3.1

constraint	comment
$x_0 = x_1$	
$x_0 = -x_1$	
$x_0 \diamond abs(x_1)$	
$x_0 = x_1 + x_2 + \ldots$	2-s complement wrap on overflow
$x_0 \leq x_1 + x_2 + \ldots$	
$x_0 \leq x_1 * x_2$	
$x_0 \geq x_1 * x_2$,	
$x_0 = \min(x_1, x_2, \ldots)$	
$x_0 = \max(x_1, x_2, \ldots)$	
$(b_0 \wedge x_0 = x_2) \vee (\neg b_0 \wedge x_0 = x_3)$	In C style this is $x_0 = (b_0?x_2 : x_3)$
$x \neq 0 \leftrightarrow y = z$	
$x_0 \in Set \leftrightarrow x_1 < x_2$	
$x_0 - x_1 \geq a_0$	
$(x_0 + a_0 \leq x_1 \vee x_1 + a_1 \geq x_0)$	
$x_0 \neq x_1$	
AllDifferent$(x_0, x_1, \ldots)$	
$[x_0, x_1 + a_0] \subseteq [a_1, a_2]$	
$[x_0, x_1 + a_0] \not\subseteq [a_1, a_2]$	
$a_0 * x_0 + a_1 * x_1 + ... \geq a_k$	Also named *weighted-sum* in [BCR05]
$(x_1 \in Set_1 \vee x_2 \in Set_2 \vee \cdots)$	Signed-clause
$c_1(x_1, \ldots, x_n) \vee c_2(x_{n+1}, \ldots, x_m) \vee \cdots$	A disjunction of any of the above.
$x_0 = x_1 * x_2$	Imprecise constraint propagation [1]
$x_0 = x_1/x_2$	Imprecise constraint propagation.
$x_0 = x_1 \% x_2$	Imprecise constraint propagation.
$x_0 = x_1^{x_2}$, i.e., $x_0 = pow(x_1, x_2)$	Imprecise constraint propagation.

Table A.1: Constraints supported by HCSP, where x_i denote variables, b_i Boolean variables, a_i constants, and $\diamond \in \{=, \leq, \geq\}$.

Chapter B

A Proof-Producing CSP Solver (A proof supplement)[1]

B.1 Introduction to the Proof Supplement

This report is meant to be a supplemental document to [VS10a]. It contains a mostly unstructured collection of things that were omitted from [VS10a] mostly due to lack of space. It includes proofs for inference rules and algorithms, introduces more inference rules, and, finally, it presents missing algorithms and improvements to algorithms that were presented in [VS10a].

B.2 Inference rules

In this section we prove the soundness of the inference rules that were introduced in Table 3 of [VS10a], and prove that it is possible to match an inference rule for any possible constraint propagation. The latter established the completeness of CSP-Analyze-Conflict .

B.2.1 Soundness proofs for the inference rules

Here we prove the soundness of the inference rules from Table 3 in [VS10a]. Throughout this section we assume that all constraints and inference rules refer to integer variables. This assumption is used only for convenience and is not a fundamental part of the work, which can be easily extended to reals.

[1] Published as a technical report in [VS10b].

Lemma B.2.1 *For an integer constant $m \in \mathbb{Z}$ and integer variables $a, b \in \mathbb{Z}$*

$$\frac{a \leq b}{(a \in (-\infty, m] \vee b \in [m+1, \infty))} \quad (LE(m)) \ .$$

Proof Combining the premise $a \leq b$ with the tautology $a \leq m \vee a > m$ yields $a \leq m \vee b > m$. Converting this to a signed clause gives the consequent of the rule $(a \in (-\infty, m] \vee b \in [m+1, \infty))$. □

Lemma B.2.2 *Assuming $V \subseteq \{v_1, \ldots, v_k\}$ such that $|V| = 1 + |D|$*

$$\frac{\textit{All-diff}(v_1, \ldots, v_k)}{(\bigvee_{v \in V} v \notin D)} \quad (AD(D, V))$$

Proof Given the premise All-diff$(v_1, \ldots, v_k)$, for $|D| = 0$, $|V| = 1$ the consequent is a tautology. Otherwise, due to counting considerations, there is no feasible assignment of $|D|$ different values to $|V| = |D| + 1$ variables, or formally $\neg(\bigwedge_{v \in V} v \in D)$. After pushing the negation into the expression, this gives $(\bigvee_{v \in V} v \notin D)$ □

Lemma B.2.3 *For an integer constant $m \in \mathbb{Z}$ and integer variables $a, b \in \mathbb{Z}$*

$$\frac{a \neq b}{(a \neq m \vee b \neq m)} \quad (Ne(m))$$

Proof Constraining the tautology $a \neq m \vee a = m$ by the premise $a \neq b$ results with $a \neq m \vee (a = m \wedge a \neq b)$ which implies $a \neq m \vee m \neq b$. Converting this to a signed clause gives the consequent of the rule $(a \neq m \vee b \neq m)$. □

Lemma B.2.4 *For an arbitrary set of values D*

$$\frac{a = b}{(a \notin D \vee b \in D)} \quad (Eq(D))$$

Proof Constraining the tautology $a \notin D \vee a \in D$ by the premise $a = b$ results with $a \notin D \vee (a \in D \wedge a = b)$ which implies the consequent of the rule $(a \notin D \vee b \in D)$. □

Lemma B.2.5 *For integer constants $m, n \in \mathbb{Z}$ and integer variables $a, b, c \in \mathbb{Z}$*

$$\frac{a \leq b + c}{(a \in (-\infty, m+n] \vee b \in [m+1, \infty) \vee c \in [n+1, \infty))} \quad (LE_+(m, n))$$

Proof Adding the premise $a \leq b + c$ to the right hand side of the tautology

$$\neg(b \leq m \wedge c \leq n) \vee (b \leq m \wedge c \leq n) \ ,$$

gives $\neg(b \leq m \wedge c \leq n) \vee a \leq m + n$. Pushing negation all the way to the literals yields $b > m \vee c > n \vee a \leq m + n$, which is equivalent to the consequent clause $(a \in (-\infty, m + n] \vee b \in [m + 1, \infty) \vee c \in [n + 1, \infty))$. □

Lemma B.2.6 *For integer constants $l_b, u_b, l_c, u_c \in \mathbb{Z}$ and integer variables $a, b, c \in \mathbb{Z}$*

$$\frac{a = b + c}{(a \in [l_b + l_c, u_b + u_c] \vee b \notin [l_b, u_b] \vee c \notin [l_c, u_c])} \quad (EQ^a_+(l_b, u_b, l_c, u_c))$$

Proof The tautology $\neg(l_b \leq b \leq u_b \wedge l_c \leq c \leq u_c) \vee (l_b \leq b \leq u_b \wedge l_c \leq c \leq u_c)$ can be rewritten as $\neg(b \in [l_b, u_b] \wedge c \in [l_c, u_c]) \vee (l_b \leq b \leq u_b \wedge l_c \leq c \leq u_c)$, which, after combining with the premise $a = b + c$, becomes

$$\neg(b \in [l_b, u_b] \wedge c \in [l_c, u_c] \vee (l_b + l_c \leq a \leq u_b + u_c) \ .$$

Writing comparisons as signed literals yields

$$\neg(b \in [l_b, u_b] \wedge c \in [l_c, u_c]) \vee a \in [l_b + l_c, u_b + u_c] \ .$$

Pushing negation all the way to the literals, and rewriting as a clause yields the consequent:

$$(b \notin [l_b, u_b] \vee c \notin [l_c, u_c]) \vee a \in [l_b + l_c, u_b + u_c]) \ .$$

□

Lemma B.2.7 *For integer constants $l_a, l_b, m, n \in \mathbb{Z}$ and integer variables $a, b \in \mathbb{Z}$*

$$\frac{\mathit{NoOverlap}(a, l_a, b, l_b)}{(a \notin [m, n + l_b - 1] \vee b \notin [n, m + l_a - 1])} \quad (\text{NO(m,n)})$$

Proof The semantics of the premise NoOverlap(a, l_a, b, l_b) is

$$\phi = (a \geq b + l_b \vee b \geq a + l_a) \ ,$$

which can be written as $\phi = (b + l_b \leq a \vee a \leq b - l_a)$. Combining ϕ with the tautology $\neg(m \leq a \leq n + l_b - 1) \vee (m \leq a \leq n + l_b - 1])$ yields

$$\neg(m \leq a \leq n + l_b - 1) \vee b + l_b \leq n + l_b - 1 \vee m \leq b - l_a) \ .$$

After simple rewriting this becomes

$$\neg(m \leq a \leq n + l_b - 1) \vee b \leq n - 1 \vee b \geq m + l_a) \ ,$$

and then $\neg(m \leq a \leq n + l_b - 1]) \vee \neg(b > n - 1 \wedge b < m + l_a)$. Rewriting with literal notation gives us the consequent of the rule:

$$(a \notin [m, n + l_b - 1] \vee b \notin [n, m + l_a - 1]) \ .$$

□

B.2.2 Completeness of inference rules

For CSP-Analyze-Conflict to be complete, it is mandatory to be able to create an explanation for any type of constraint, including for explicit table relations. When an unsatisfiability proof is printed, the proof derives each explanation from a constraint using an inference rule. For the proof system to be complete, we must be able to match an inference rule for any implication. Like in the proof, CSP-Analyze-Conflict may also use inference rules to derive explanations.

In order to discuss inference rules and their consequent explanations, we need to focus on valid implications first. The following definition summarizes what we refer to as a propagator. These are weak requirement that are satisfied by any propagator that we are aware of.

Definition B.2.1 (Constraint propagator assumption) *Every constraint $r \in \mathcal{C}$ is associated with a propagator by the same name. We assume that the constraint propagator $r(v_1, \ldots, v_k)$ meets the following criteria:*

- *when a complete assignment to $v_1, \ldots, v_k$ violates the constraint $r(v_1, \ldots, v_k)$ the propagator will recognize it;*
- *if it modifies a domain from D_i to D'_i then $D'_i \subseteq D_i$ and the values in $D_i \setminus D'_i$ are not supported by r (in other words, for every assignment $v_i \leftarrow x_i$ such that $x_i \in D_i \setminus D'_i$, there is no assignment to $v_1, \ldots, v_k$ from their respective domains that satisfies the constraint r); and*
- *it terminates in finite time.*

Note that this assumption leaves the freedom to choose the strength of the propagator. Some known strategies are *arc consistency*, which is the strongest, *bounds consistency*, and pure *backtrack-search* algorithm, which is the weakest.

Recall that CSP-Analyze-Conflict requires an explanation of inferred literals, and hence we need to show that such an explanation can always be given. For this purpose we define a generic inference rule and then prove that it is applicable to all constraints.

Definition B.2.2 (The generic inference rule) *Consider $r(v_1, \ldots, v_k)$, an arbitrary constraint, which when combined with literals $v_1 \in D_1, \ldots, v_k \in D_k$ implies literal $v_i \in D'_i$. The generic, parameterized, implication rule $G_{r,i}(A_1, \ldots, A_k)$, as defined below, generates an explanation for this implication*

$$\frac{r}{(v_1 \in A_1 \vee \cdots \vee v_k \in A_k)} \quad (G_{r,i}(A_1, \ldots, A_k)) \ ,$$

where for each $j = 1, \ldots, i-1, i+1, \ldots k$, *we define* $A_j = \overline{D_j}$ *and where* $A_i = D'_i \cup \overline{D_i}$

We now show that Definition B.2.2 gives a valid inference rule for an arbitrary constraint.

Lemma B.2.8 (Existence of an inference rule) *Let* u *be a node in an implication graph that represents a literal that was implied by an arbitrary constraint* r. *There exists an inference rule that its premise is* r *and its consequent is an explanation of* u.

Proof For any constraint $r(v_1, \ldots, v_k)$, Definition B.2.2 provides the generic propagation rule $G_{r,i}(A_1, \ldots, A_k)$, which generates the following explanation for an implication of $v_i \in D'_i$:

$$c = (v_1 \in \overline{D_1} \vee \cdots \vee v_{i-1} \in \overline{D_{i-1}} \vee v_i \in (D'_i \cup \overline{D_i}) \vee v_{i+1} \in \overline{D_{i+1}} \vee \cdots \vee v_k \in \overline{D_k}) \ ,$$

which can also be written as:

$$c = (v_1 \notin D_1 \vee \cdots \vee v_k \in D_k \vee v_i \in D'_i) \ .$$

To prove that the clause c is indeed an explanation, recall, we have to show that, for $l_1 = (v_1 \in D_1), \ldots, l_k = (v_k \in D_k)$:

1. $r \rightarrow c$,

2. $(l_1 \wedge \cdots \wedge l_k \wedge c) \rightarrow l$.

To prove the first property we will show that every possible assignment $x_1, \ldots, x_k$ satisfies $\neg c(x_1, \ldots, x_k) \rightarrow \neg r(x_1, \ldots, x_k)$. The clause c is falsified only if all its literals are *false*, i.e., $v_1 \in D_1 \wedge \cdots \wedge v_k \in D_k \wedge v_i \notin D'_i$. In other words, all values are taken from their respective domains D_j and $x_i \in D_i \setminus D'_i$, which is one of the values removed by the propagator r. According to Definition B.2.1, r will not remove x_i from D_i if $x_1 \in D_1 \wedge \cdots \wedge x_k \in D_k$ satisfies $r(x_1, \ldots, x_k)$, implying that $\neg r(x_1, \ldots, x_k)$. The conclusion is that $\neg r(x_1, \ldots, x_k)$ whenever $\neg c(x_1, \ldots, x_k)$.

For the second property we analyze all assignments $v_1 = x_1 \wedge \cdots v_k = x_k$ which satisfy the left side of the implication. Such an assignment has to satisfy $x_1 \in D_1 \wedge \cdots \wedge x_k \in D_k$, in which case all literals $v_j \notin D_j$ of c are falsified except for $v_i \in D'_i$, i.e., l. This shows that an assignment that satisfies the left-hand side of the implication also satisfies its right-hand side $\square$

B.3 Algorithms

B.3.1 CSP-Analyze-Conflict algorithm, the proof

First, we repeat Algorithm 2.1 which appears in [VS10a]:

```
1: function CSP-Analyze-Conflict
2:     cl:= Explain (conflict-node);
3:     pred:= Predecessors (conflict-node);
4:     front:= Relevant (pred, cl);
5:     while (¬Stop-criterion-met (front)) do
6:         curr-node:= Last-node (front);
7:         front:= front\curr-node;
8:         expl:= Explain (curr-node);
9:         cl:= Resolve (cl, expl, var(lit(curr-node)));
10:        pred:= Predecessors (curr-node);
11:        front:= Distinct (Relevant (front ∪ pred, cl));
12:    end while
13:    add-clause-to-database(cl);
14:    return clause-asserting-level(cl);
15: end function
```

To prove the correctness of Algorithm 2.1, we first prove the loop invariant that is mentioned briefly in [VS10a].

Lemma B.3.1 (CSP-Analyze-Conflict **loop invariant)** *The following invariant holds just before line 6: The clause cl is inconsistent with the labels in* front*.*

Proof Consider the conflicting constraint p, i.e., the constraint that labels the edges leading to *conflict-node*. On the first iteration, the literals of *pred* conflict the constraint p . Because *cl* is the explanation clause of p it also conflicts on the conjunction of *pred*. Because Relevant at line 4 keeps the nodes which relevant to *cl*, i.e., share the same variables, then the labels of *front* also conflict *cl*.

Assuming that the invariant holds on iteration $n-1$, we will show that the invariant holds at the n-th iteration, if executed. For this proof we denote by *cl* and cl' the values of *cl* before and after the update at line 9, respectively, and similarly we use *front* and *front'*. Except for the literal with $v = var(lit(curr\text{-}node))$, according to the definition of *expl*, all other literals are falsified by *pred*. cl' has three types of literals: the first is from *expl* and do not refer to v, the second is from *cl* and do not refer to v , and the third refers to v and contains a conjunction of literals from the former two sources.

1. The literals from cl are falsified by the conjunction of $lits(front)$ and hence also by the conjunction of $lits(front \cup pred)$.

2. For an implication going from literals $\{l_1, \ldots, l_k\} = lits(pred)$ to literal $l = lit(curr\text{-}node)$, recall, the second requirement from an explanation c is $(l_1 \wedge \cdots \wedge l_k \wedge c) \rightarrow l$. In order for c to be able to imply a literal l, as required, all literals of cl must be falsified by $l_1 \wedge \cdots \wedge l_k$, except for the literal that constrains v. Because $l_1 \wedge \cdots \wedge l_k$ is the conjunction of $lits(pred)$, then literals cl that do not constrain v, which are of the second type of literals, are also falsified by the conjunction of $lits(front \cup pred)$.

3. We want to prove that the literal of cl' labeled with v is falsified by the $lits(front')$. First we introduce the following naming convention: ω_v is the disjunction of all literals of clause ω which affect v, and similarly N_v is the conjunction $\bigwedge_{l \in lits(N) | var(l)=v} l$. We also reuse the notation used in the definition of explanations where l is the literal of *curr-node* and $l_1, \ldots, l_k$ are the literals of *pred*.

   The claim to be proven can now be reformulated as $(cl'_v \wedge lits(front')) = false$. To prove this, first consider the invariant from iteration $n-1$ which gives $(cl_v \wedge l) = false$, or as a clause:

   $$(\neg cl_v \vee \neg l) \ .$$

   The definition of explanations states that $(l_1 \wedge \cdots \wedge l_k \wedge expl) \rightarrow l$, or as a clause:

   $$(\neg l_1 \vee \cdots \vee \neg l_k \vee \neg expl \vee l) \ .$$

   Resolving the last two clauses with v as pivot results with:

   $$(\neg l_1 \vee \cdots \vee \neg l_k \vee \neg expl \vee \neg cl_v) \ ,$$

   which is the same as

   $$\neg((l_1 \wedge \cdots \wedge l_k \wedge) expl \wedge cl_v)) \ ,$$

   which shows that $expl \wedge cl_v$ is falsified by $lits(pred)$, and hence $cl'_v = expl_v \wedge cl_v$ is also falsified by $lits(front \cup pred)$.

   The call to RELEVANT at line 11 removes only nodes which are irrelevant to the falsification of cl', hence, it is left to show that *front*, produced by DISTINCT at line 11, also falsifies cl'_v. For this part of the proof we need some formalism first. We depict $n_1, \ldots, n_q$ as the nodes of $lits(front \cup pred)$ for which $var(lit(n_j)) = v$ according to the propagation order that created them. Using this formalism we can say that DISTINCT will remove $n_1, \ldots, n_{q-1}$ and keep only n_q, we need to show that this node is sufficient to falsify cl'_v. Because how propagators are allowed to work, literals of succeeding nodes must refer to decreasing domain sized, and hence

$n_q \rightarrow n_j$ for each $j = 1, \ldots, q$. This result means that all nodes $n_1, \ldots, n_{q-1}$ are redundant, such that if $(lit(n_1) \wedge \cdots \wedge lit(n_q) \wedge cl'_v) = \mathit{false}$ then it is also true that $(lit(n_q) \wedge cl'_v) = \mathit{false}$. Because cl'_v was falsified by the literals of $n_1, \ldots, n_q$, as shown above, then it must be falsified by n_q which is entered into *front*'.

As we seen, all literals of the new clause cl' are falsified by the literals of *front*'. This shows that iteration n also satisfies the loop invariant. □

Theorem B.3.1 (Algorithm 2.1 correctness) CSP-ANALYZE-CONFLICT *, i.e., Algorithm 2.1, is sound and complete:*

- *soundness – the returned clause cl is derived from the CSP ϕ such that $\phi \rightarrow cl$, and it is either falsified at the target decision level or is an asserting clause that will cause propagation at clause-asserting-level(cl).*
- *completeness – the algorithm terminates and returns cl.*

Also, CSP-ANALYZE-CONFLICT *returns a target level which forces backtrack of at least one level.*

Note that this theorem allows the resulting *cl* to be *false* even at the target of backtrack. This situation happens often in practice and can be easily eliminated by a small modification to the algorithm, as described in Section B.4.

Proof Soundness – The requirement of $\phi \rightarrow cl$, is trivial because *cl* is created through a set of proven inference rules from premises and through applications of the resolution rule. The requirement of *cl* to be an asserting clause is met through the STOP-CRITERION-MET which is true in two cases:

- *front* nodes have no predecessors, i.e., they refer to initial domains. According to the loop invariant (Lemma B.3.1) all literals of *cl* are falsified by the conjunction of the literals of *front*, meaning that *cl* is *false* even with the initial domains.
- The latest node of *front* is the only one from decision level d and the next latest node is from decision level $d' < d$, such that function clause-asserting-level(cl) will return this d'. Backtracking will undo all decisions and implications done from $d'+1$ and later a target state at which, we need to show, *cl* will be an asserting clause. At the target state all nodes of *front*, except for the latest one, are still present, which according to Lemma B.3.1 must falsify all literals except for the latest one. This means that *cl* is either falsified or an asserting clause at the end of decision level d'.

Completeness – First we show that the preconditions of all auxiliary functions are fulfilled allowing them to successfully terminate. Assuming that CSP-Analyze-Conflict is passed a conflict-node then the first Predecessors must succeed. The first Explain must succeed because it is possible to generate an explanation for the conflicting constraint (Lemma B.2.8). Inside the loop ¬Stop-criterion-met makes sure that *curr-node* has predecessors making predecessors succeed. Like with the first Explain the one in the loop must also find an explanation. All other used algorithms do not have special assumptions or preconditions.

The algorithm must terminate because each iteration removes the latest node, and inserts earlier nodes of *pred* instead of it, coupled with the fact that the implication graph is a DAG, each iteration is guaranteed to have *curr-node* which is earlier than at the previous iteration. Because the implication graph is finite then the number of iteration must be finite.

Backtrack level – as shown above, the target level is d' which is smaller than d which itself the current decision level, or earlier. This means that the caller will be instructed to backtrack to at least one decision level back. □

B.4 Enhancements and optimizations

In this section we describe several enhancements to the definitions and to the algorithms that may either improve performance or minimize proof size. At the present time we do not have exact measurements of the effectiveness of these modifications. Nevertheless these enhancements have interesting properties, making them worth mentioning.

B.4.1 CSP-Analyze-Conflict node rejuvenation

A problem that the proof of CSP-Analyze-Conflict (Theorem B.3.1) showed is that a conflict clause may be conflicting immediately after backtrack, i.e., with no new decision. This problem is solved in HCSP by the following, trivial, modification to the algorithm.

At lines 4 and 11 Distinct is called. After this call we add a call to a new function Rejuvenate (*cl*,*front*), which finds the earliest nodes that still falsify *cl*. This modification preserves the invariant of the loop because it moves only to such earlier nodes that still falsify *cl*. The proof of Theorem B.3.1 with the amended algorithm is mostly unchanged since it relies on the invariant, which is not affected by the modification.

This rejuvenating can have a positive impact on conflict clause and proof sizes as it allows CSP-Analyze-Conflict to ignore all implications done between the original node and the rejuvenated node. However, it may also have a negative effect since by

ignoring an implication we may lose a good candidate for clause resolution that would erase the literal from cl and reach a UIP. Instead, by ignoring a good candidate for resolution, we will have to apply resolution many times before a UIP is reached.

B.4.2 Augmented explanations

Let $(l_1, l), \ldots, (l_n, l)$ be the incoming edges of a node u such that $lit(u) = l$. If c is an explanation clause of u, recall, then $(l_1 \wedge \cdots \wedge l_n \wedge c) \rightarrow l$. We now propose to weaken this requirement.

Definition B.4.1 (augmented explanations) *Let u be a node in the implication graph such that $lit(u) = l$. Let $(l_1, l) \ldots (l_n, l)$ be the incoming edges of u, all of which are labeled with a constraint r. Let l' be the literal in the clause cl just before the resolution step in* CSP-Analyze-Conflict *, such that $var(l) = var(l')$. A signed clause c' is an* augmented explanation clause *of a node u and a clause cl if it satisfies:*

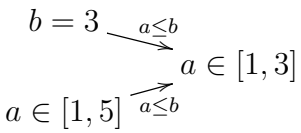
1. $r \rightarrow c'$,
2. $(l_1 \wedge \cdots \wedge l_n \wedge c') \rightarrow \neg l'$.

This definition lets us find c' which is not an explanation clause according to the original definition, but which is sufficient for our purpose, since while it is still implied by the original constraint r, it also holds that

$$\text{Resolve}(cl, c', var(l)) \rightarrow \text{Resolve}(cl, c, var(l)) \ .$$

In other words, we derive a stronger resolvent. This may lead to shorter proofs down the line.

Example B.4.1 *The literals $l_1 = (b = 3)$, $l_2 = (a \in [1, 5])$ together with the constraint $r = (a \leq b)$ imply $l = (a \in [1, 3])$. This implication is depicted in the following small diagram:*



The only valid explanation clause is derived using $LE(3)$:

$$c = (a \in (-\infty, 3] \vee b \in [4, \infty)) \ .$$

Indeed $(l_1 \wedge l_2 \wedge c) \rightarrow l$.

Now suppose that $cl = (a = 5 \vee c = 1)$ *and hence* $l' = (a = 5)$*. We need to find* c' *such that* $(l_1 \wedge l_2 \wedge c') \rightarrow \neg l'$*. Although* c *is an explanation clause, we can get a stronger such clause with* $LE(4)$*:*

$$c' = (a \in (-\infty, 4] \vee b \in [5, \infty)) \ ,$$

which is sufficient. Resolution of cl *with* c' *results in* $(b \in [5, \infty) \vee c = 1)$ *whereas resolving with* c *would result in the weaker clause* $(b \in [4, \infty) \vee c = 1)$*.*

What happens in this example is that c' *does not have to eliminate the value 4 from the domain of* a *because it is allowed by* $\neg l'$*. As a result the other literal, referring to* b*, becomes stronger.* □

B.4.3 Lookahead explanations

The weaker definition of *augmented explanations* gives us more freedom to choose a clause from a bigger set of possible clauses. Some freedom is also present in regular explanation clauses, especially when explaining a conflicting constraint. For example, the constraint $a \leq b$ is conflicting for domains $D_a = [10, 20]$ and $D_b = [0, 8]$, for which both $c = (a \in (-\infty, 8] \vee b \in [9, \infty))$ and $c = (a \in (-\infty, 9] \vee b \in [10, \infty))$ are valid explanations.

We have discovered that using this freedom wisely can considerably shorten both proof size and run-time. If *curr-node* is associated with variable v, we simply prefer an explanation which has a stronger literal associated with the rightmost node of *predecessors* which is not associated with v. The effect of this preference is twofold:

1. At the next iteration of CSP-Analyze-Conflict , there is a bigger chance that this literal l' will be a pivot than any other literal of the explanation clause. Making this literal smaller may strengthen the next augmented explanation because it will weaken the right-hand side of the implication in the requirement of *augmented explanations*:

   $$(l_1 \wedge \cdots \wedge l_n \wedge c') \rightarrow \neg l' \ ,$$

   which will give more freedom for constructing c'.

2. By producing a stronger literal, there is a bigger chance that Rejuvenate will rejuvenate the node of $var(l')$ to a previous decision level, saving the need for some resolutions and, eventually, facilitate the creation of a smaller conflict clause.

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- **ניתוחי סתירות שאינם מבוססי פסוקיות.**
תוך שימוש בכלל ההיסק Combine פותר האילוצים HCSP מפעיל אלגוריתם חדשני לבחינת סתירות אשר לומד אילוצים מורכבים. הוא מסוגל, לדוגמא, לשלב את האילוץ $x_1 + 2x_2 + 7x_3 \geq 0$ עם $x_1 + 3x_2 \geq 5$ ולקבל $2x_1 + 5x_2 + 7x_3 \geq 5$, ואז לשלב את התוצאה עם $2x_1 + 5x_2 + 7x_3 \leq 4$ ולזהות את הסתירה ללא צורך בהחלטות נוספות. שיטות בחינת סתירות אחרות כגון [BK, KB05, VS10a] דורשות סתירות של פסוקיות נלמדות כדי להגיע למסקנה דומה.

עקב השימוש בכללי היסק שמתמחים בשילובים מסויימים של זוגות אילוצים, HCSP מזכיר במעט פותרי בעיות SMT. פותרי בעיות SMT מקבצים אילוצים ל-תיאוריות, אשר מעובדות באופן בלתי-תלוי. מרבית ההתממשקות בין האילוצים מבוצעת בתוך כל תיאוריה בנפרד, באופן שמעט מזכיר את כללי היסק ה-combine שפועלים רק על טיפוסי זוגות מסויימים של אילוצים. אבל לא כמו ב-HCSP, מערכת ההיסקים של SMT מוגבלת בעיקר לקיבוע של אילוצים קיימים תוך שימוש בבחינת הסתירות ובהפצת התיאוריה (Theory Propagation), ולא על ידי למידת אילוצים חדשים לחלוטין בזמן בחינת הסתירות.

הבדל משמעותי נוסף בין SMT לבין HCSP הוא מערכת ביצוע ההחלטות וההפצה. ב-SMT החלטות מתבצעות על ערכים בוליאניים אשר מקבעים אילוצים או מבטלים אותם כך שהבעיה המלאה תסתפק. לעומת זאת ב-HCSP ההחלטות מתבצעות על כל המשתנים. ההפצה של התאוריות ב-SMT יכולה ללמוד פסוקיות חדשות שמשפיעות על אילוצים קיימים. ב-HCSP ההפצה משנה את כלל המשתנים ורק בחינת הסתירות יכולה ללמוד אילוצים חדשים, כולל אילוצים בוליאנים אשר משפיעים על קיבוע של אילוצים קיימים.

- **הוכחת אי ספיקות.**
סעיף 2.4 מתאר כיצד HCSP מייצר הוכחות אי-ספיקות, הניתנות לבדיקת מכונה, תוך שימוש בכללי היסק בסיסיים. כללי ההסבר (explanation rules) משמשים לי-יצור פסוקיות מרובות ערכים (signed-clauses) מתוך האילוצים, ועליהן פותר האילוצים מפעיל את כלל ה-resolution כדי לקבל פסוקיות חדשות. לפי מיטב ידיעתנו HCSP הוא פותר האילוצים היחיד אשר מייצר הוכחות, תכונה אשר עד כה היתה קיימת רק בפותרי SAT.

- **אינטרפולציה.**
HCSP יכול ליצור אינרפולנטים, וככל הידוע לנו הוא פותר האילוצים היחיד המסוגל לכך. בסעיף 4.5 אנו מוכיחים את נכונות אלגוריתם 4.4, אשר מוצא את האינטרפולנט של קרייג עבור בעיית האילוצים. האלגוריתם מבוסס על אלגוריתם דומה הנמצא בשימוש חלק מפותרי ה-SAT עבור נוסחאות עם פסוקיות ותוך שימוש בכלל ה-binary resolution.

H.3.4 תרומת המחקר

התיזה בנויה כאסופת המאמרים המורחבים: [VS10a] אשר פורסם ב-AAAI'10, המאמר [VS14] שנמצא בתהליך הגשה, ותוסף הוכחות אשר פורסם כדו"ח טכני [VS10b]. פרק Chapter 4 מכיל חומר נוסף אשר לא מופיע בפירסומים אלו. התרומות המרכזיות של המחקר הזה מסוכמות ברשימה הבאה:

- פותר אילוצים מהיר.
HCSP הוא פותר אילוצים חדש שנכתב כחלק ממחקר זה. הוא פותר האילוצים המהיר ביותר[4] אשר מכיר את שפת הקלט של תחרות פתרון האילוצים [DLR09] של 2009, האחרונה אשר משתמשת בשפת קלט זו. מעבר לשפת הקלט `MiniZinc` החדשה יותר מראה תוצאות מבטיחות כאשר משווים מול פותרי האילוצים המכירים שפה זו.

- הסברת אילוצים עצלה.
בפרק 2 מופיע הסבר איך HCSP מעדן את המושגים הבסיסים המופיעים ב- [BK], [KB05], ו-[OSC09]. האלגוריתמים האלו מסבירים ואז מבצעים כל הפצה של כל אילוץ במונחים של פסוקיות. לעומתם HCSP מבצע הסברת אילוצים עצלה, כלומר רק בזמן ניתוח הסתירה ולא בזמן הפצת האילוצים. דחיית ההסברה לזמן ניתוח הסתירה חוסך בזיכרון, על ידי הקטנת מספר האילוצים שצריך להסביר. בנוסף, שיטה זו מאפשרת בחירת הסבר טוב יותר בזמן ניתוח הסתירה, כאשר יש מספר הסברים אפשריים. יתרה מכך, דחיית יצירת ההסבר מאפשרת לייצר הסבר מותאם, כמוסבר בסעיף B.4.2. הסבר מותאם לא מסביר את ההפצה של האילוץ, אלא מסביר למה האילוץ מוביל לסתירה.

- כלל ההיסק **Combine**
אלגוריתם 3.1 שמופיע בסעיף 3.2.2 מציג את צירוף האילוצים (*constraint combination*) אשר מרחיב את binary resolution של SAT רב-ערכי. בסעיף 3.4.1 מוצג כלל ההיסק *combine*, עבור האילוצים c_1, c_2 והציר x (pivot באנגלית), המוגדר באופן הבא:

$$\frac{c_1(x,\bar{y}) \qquad c_2(x,\bar{y})}{x \notin \mathcal{X} \vee \exists x' \in \mathcal{X}.[c_1(x',\bar{y}) \wedge c_2(x',\bar{y})]} \quad \text{(combine(x))} \quad .$$

כאשר $\bar{y}$ הוא אוסף המשתנים המאולצים לפחות עלי ידי אחד מהאילוצים ו- $D_{\bar{y}}$ הוא התחום שלהם. ובנוסף:

$$\mathcal{X} = \{x' \in \mathcal{Z} | \forall \bar{y'} \in D_{\bar{y}}.[\neg c_1(x',\bar{y'}) \vee \neg c_2(x',\bar{y'})]\} \quad .$$

המחקר מראה שלכלל ההיסק *combine* יש תכונות אשר גורמות לו לתפקד כראוי בזמן ניתוחי הסתירות. בנוסף, מספר כללי היסק חזקים נגזרים מכלל ההיסק הכללי.

[4] HCSP הוא המהיר ביותר אם מתעלמים מאילוצים טבלאיים. המימוש הנוכחי של אילוץ הטבלה אינו יעיל ויש לכתבו מחדש.

בעיות מסוג CNF SAT. וגם כבעיה אקספוננציאלית עבור MAC כאשר אילוצי אי-השוויון בלתי תלויים כי על האלגוריתם לבדוק $(n-1)!$ השמות לפני שהוא מבין שאין פתרון לבעיה.

אלגוריתם ה-MAC לא לומד מהחלטות כושלות ולכן חוזר על ניסיונות השמה דומים אשר מביאים אותו לסתירה פעם אחר פעם. גם בדוגמא האחרונה ראינו שהאלגוריתם ניסה החליט $x = 1$, הגיע לסתירה, ומייד ניסה את $x = 2$ שהוביל לסתירה דומה. HCSP רוצה להמנע מחזרה על נסיונות כושלים על ידי למידת הסיבות האמיתיות לסתירות. בבעיה האחרונה היינו רוצים ש-HCSP יזהה את עקרון שובך היונים שאומר שאין פתרון לבעיה הזו גם בלי לעבור על כל הערכים.

H.3.3 ניתוחי הסתירות

תהליך פתרון בעיית האילוצים כולל ביצוע ניסיונות החלטה שגויים, שמובילים לסתירה, וחזרה מהם (backtracking). בעבר היו מספר הצעות לאלגוריתמים שיכלו להמנע מחזרה על החלטות שגויות, על ידי ניתוח הסתירות, אך חלקם היו חלשים מכדי לזהות החלטות שהיו דומות להחלטות שגויות ואילו לחלקם זמן הביצוע היה לא יעיל. לאחרונה גם פורסמו אלגוריתמים [OSC09] שמנתחים את הסתירות תוך שימוש בתוכנה לפתרון בעיות ספיקות בתחשיב הפסוקים (a SAT solver).

לעומת שיטות אלו HCSP מנתח את הסתירות ישירות מעל בעיית האילוצים ולומד אילוץ חדש אשר מונע הרבה החלטות שגויות דומות. ניקח לדוגמא את הבעיה הבאה:

$$z = x + y, x = y, D_x = D_y = D_z = \{0, 1, \cdots, 6\}$$

אם התבצעה ההחלטה, השגויה, $z = 1$ אז $z = x + y$ יצמצם את התחומים ל-$D_x = D_y = \{1, 2, \cdots, 6\}$. ל-$x = y$ לא תהיה השפעה על התחומים וזאת למרות שאין פתרון תחת ההחלטה הראשונה, לכן מבצעים החלטה נוספת $x = 1$ ואז הסתירה מתגלית.

בדוגמא זו, ללא ניתוח סתירה יש צורך לבצע החלטות עם כל ערכי D_x אשר מובילים לסתירה דומה. ניתוח הסתירה, תוך שימוש באילוצים ובהנחה שיש כלל היסק מתאים, יכול לזהות ש-$z = 2 * x$ נובע מהאילוצים[3]. האילוץ החדש סותר את ההחלטה הראשונה $z = 1$ ולא תלוי בהחלטות של x ולכן חוסך את הצורך לבצע הרבה החלטות עבור x. בנוסף, האילוץ $z = 2 * x$ מצמצם את התחומים:

$$D_x = D_y = \{1, 2, 3\}, D_z = \{2, 4, 6\}$$

צימצום זה מונע החלטות שגויות שלכאורה אינן קשורות להחלטה השגויה המקורית של $z = 1$. למשל, $x = 4$ נאסר בגלל תחום הערכים המצומצם של z.

[3] השילוב $x = y$ ו-$z = x + y$ הוא נוח לצורך הדגמה של האלגוריתם אך כלל היסק שיוצר את $z = 2 * x$ לא מומש עדיין ב-HCSP, למרות שאין זה מסובך לממשו.

ושוב. שיטה זו של צימצום תחום המשתנים נקראת הפצה (propagation) והיא נדבך מרכזי ברוב פותרי האילוצים בכלל וב-HCSP בפרט.

H.3.2 ביצוע החלטות

ההפצה אומנם מצמצמת את מרחב החיפוש בצורה משמעותית אך לא די בכך. נבחן לדוגמא את הבעיה הבאה:

$$x = y * z, x = y, D_x = D_y = D_z = \{1, 2, 3, 4\}$$

הפצת האילוץ $x = y$ לא משפיעה על התחומים כי עבור כל ערך אפשרי של x קיים ערך מתאים של y המספק את האילוץ. באותו האופן ניתן לראות שהפצת האילוץ $x = y * z$ לא משפיעה על התחומים, שכן עבור כל ערך של x ההצבות $y = 1$ וגם $x = z$ מקיימות את האילוץ. למרות שהפצת כל אילוץ בנפרד לא מצמצמת את התחומים, רוב צירופי הערכים לא עומדים בדרישות (למשל $x = y = z = 4$ מספק את האילוץ $x = y$ אך לא את $x = y * z$).

לכן פרט להפצת האילוצים יש לנסות ולהציב ערכים שונים למשתנים באופן הבא. פותר האילוצים מבצע החלטה שרירותית ומציב ערך לאחד המשתנים. ננסה למשל $x = 4$ ואז נפיץ את $x = y*z$ אשר מצמצם את התחומים ל-$D_z = D_y = \{1, 2, 4\}$. כשמפיצים את $x = y$ אז מקבלים $D_y = \{4\}$ ושוב מפיצים את $x = y * z$ שמצמצם את התחום החופשי האחרון ל-$D_z = \{1\}$. בשלב זה כל התחומים מכילים ערך בודד כך ש-$x = 4, y = 4, z = 1$. בחינת האילוצים מראה שהערכים האלו אכן מקיימים את האילוצים $x = y, x = y * z$.

שילוב כזה של הפצת אילוצים והחלטות שבוחרות ערכים למשתנים פורסם ב-[Mac77] תחת השם MAC (Maintain Arc Consistency). תיאור מלא של MAC מופיע ב-Algorithm 1.1. יש להדגיש שבעיית אילוצים סבירה חלק מנסיונות ההצבה לא מובילים לפתרון ויש צורך לחזור ולנסות ערכים שונים. במקרה הגרוע ביותר יש צורך לנסות את כל הערכים כדי להבין שאין פתרון לבעיה. למשל לפי עקרון שובך היונים לבעיה הבאה אין פתרון:

$$x \neq y, y \neq z, z \neq x, D_x = D_y = D_z = \{1, 2\}$$

בבעיה זו תחומי המשתנים לא מושפעים מהפצת כל אילוץ בנפרד ולכן מנסים להציב $x = 1$. אילוץ $x \neq y$ מפיץ את $D_y = \{2\}$ ואילו $x \neq z$ את $D_z = \{2\}$. לבסוף $y \neq z$ מגלה את הסתירה. בגלל שביצענו השמה ניסיונית של $x = 1$, שנכשלה, האלגוריתם חוזר אחורה למצב שלפני ההשמה, קרי $D_x = D_y = D_z = \{1, 2\}$. אז מנסים את הערך האחר של D_x ומציבים $x = 2$. ערך זה, כמו קודמו, מוביל לסתירה ומכיוון שניסינו את כל האפשרויות של x נותר להסיק שאין פתרון לבעיה.

ניתן להגדיל את הבעיה ל-n כלשהו של משתנים, עם תחומי ערכים $D_{v_i} = \{1, 2, \ldots, n - 1\}$ ואילוץ אי-שוויון בין כל זוג משתנים. בעיה זו ידועה כבעיה אקספוננציאלית עבור פותרי

5 חדרים שפועלים 40 שעות כל אחד יש $5 \times 40 = 200$ משתנים. לכן יש "רק" $100^{200} = 10^{400}$ השמות אפשריות (תוך התעלמות מהאילוצים). גם אם היינו בודקים מיליארד השמות בשנייה על מיליון מחשבים היה עלינו להמתין בסבלנות מעט יותר מ- 10^{177} שנה. זהו אינו זמן סביר עבור בני התמותה.

H.2 פותר האילוצים

כחלק מהמחקר פותח פותר אילוצים בשם HaifaCSP או בקיצור HCSP אשר מסוגל לפתור בעיות מסובכות עם מספר רב של אילוצים מטיפוסים שונים. לכתיבתו נדרשו $60,000$ שורות C++ ללא הסתמכות על קוד חיצוני כמעט. HCSP מבין 3 שפות אילוצים תיקניות שונות:

- שפת XCSP-2.1 שהוגדרה ב- [RL09].
- שפת `MiniZinc` כפי שהוגדרה ע"י [MSKS10].
- חלק משפת OPB המאפשרת קלט מסוק פסאודו-בוליאני [MR+06].

H.3 האלגוריתם לפתרון האילוצים

H.3.1 הפצת האילוצים

מעשית לא ניתן לפתור את בעיית האילוצים על ידי בחינת כל הצירופים ולכן פותר האילוצים משתמש בשיטות שמלכתחילה מונעות ממנו את הצורך לבחון צירופים לא סבירים. שיטה מקובלת לצימצום מרחב החיפוש נקראת הפצה propagation ופורסמה ב-[Mac77]. כדי להמחיש את השיטה נבחן את הדוגמא שבה יש אילוץ בודד $x < y$ ותחום הערכים החוקיים של המשתנים הוא $x, y \in \{1, 2, 3\}$. ברור שאם נבחר $x = 3$ אז $x = 3 < y$ ולכן לא נוכל למצוא $y \in \{1, 2, 3\}$ שיספק את האילוץ. באותו האופן אם נבחר $y = 1$ אז לא נוכל למצוא x חוקי שיקיים את האילוץ. לכן, עוד לפני שמנסים להציב ערכים למשתנים, נוכל לעדכן את תחום הערכים החוקיים באופן הבא: $x \in \{1, 2\}$ וגם $y \in \{2, 3\}$.

נניח שגם נתון האילוץ $y < x$ ונניח לצורך הפשטות שפותר האילוצים לא יודע להסיק ש-$x < y$ ו-$y < x$ סותרים זה את זה. בגלל האילוץ $y < x$ והתחומים $x \in \{1, 2\}, y \in \{2, 3\}$ ברור שאין פתרון לבעיית האילוצים $x < y, y < x$. יש לשים לב שהראינו שאין פתרון עבור התחומים ההתחלתיים $x, y \in \{1, 2, 3\}$ ולא הראינו זאת עבור תחומים שונים. למשל, פעולות אלו לא הראו שאין פתרון עבור $x, y \in \{1, 2, 3, 4, 5, 6\}$. אם נחזור ונבחן את שני האילוצים לסירוגין נוכל להיווכח שגם בעיה זו אינה פתירה. ועבור תחומים אחרים ניצטרך לחזור על התהליך שוב

תקציר

H.1 בעיית פתרון האילוצים (CSP)

ניתן לתאר בעיות הנדסיות רבות כאוסף החלטות אשר צריכות לספק קבוצת אילוצים. כאשר בוחרים צוות טיסה יש לקחת בחשבון שמטוס נוסעים יטוס, למשל, עם טייס ושלשה מסייעים ושהם לא מצוותים לטיסה אחרת באותו הזמן. כאשר בוחרים נתיב נסיעה של רכב יש אילוצים על המסלולים האפשריים ומהירות הנסיעה. בהרבה מקרים ישנם יחסי גומלין מורכבים בין אילוצים. נבחן לדוגמא את בעיית שיבוץ ההרצאות באוניברסיטה. מערכת שעות ההרצאות מאולצת על ידי העדפות המרצים, חדרים שיכולים להכיל לכל היותר הרצאה אחת ברגע נתון, סטודנטים שיכולים להמצא רק במקום אחד באותה השעה, דרישות תכנית הלימודים, ועוד. ישנן בעיות רבות נוספות המערבות אילוצים, כגון מיקום מכולות על ספינה, ניהול תלויות בין תוכנות מחשב, תצורת רכבים, ועוד.

הבעיה של מציאת פתרון המקיים את כל האילוצים נקראת בעית פתרון האילוצים, או באנגלית constraint Satisfaction Problem - CSP ובקיצור CSP. בעית פתרון אילוצים טיפוסית היא קשה לפתרון לא רק עקב היותה NP-קשה אלא גם מפאת גודלה. סוגיית הסיבוכיות נידונה בהרחבה בסעיף 1.1.1. כדי להמחיש את סוגיית הסיבוכיות נבחן את אחת האפשרויות לייצוג בעיית שיבוץ ההרצאות באוניברסיטה. למשל, נרצה לדעת איזו הרצאה יש לשבץ ביום ראשון בשעה 9:00 בחדר 315. לפיכך נגדיר משתנה בעייה $v_{315,1,9}$ שיכיל ערך מספרי שיגדיר את פרטי המרצה בשעה הנתונה. המשתנה יכול לקבל ערך מתחום מוגבל - אוסף המרצים האפשרי[2]. בנוסף למשתנים והתחום אליו הם מוגבלים, הבעיה מכילה אילוצים רבים כדוגמת $v_{315,1,9} \neq v_{314,1,9}$. אילוץ זה אומר שנאסר על המרצה, שמאד אוהב לעשות זאת, להרצות ב-9:00 ביום ראשון גם בחדר 315 וגם 314 בו זמנית.

באופן פורמלי, בעיית האילוצים מוגדרת על ידי השלשה $\langle \mathcal{C}, \mathcal{V}, \mathcal{D} \rangle$ כאשר $\mathcal{V}$ הם המשתנים, $\mathcal{D}$ מגדיר לכל משתנה את תחומו, ו-$\mathcal{C}$ הוא אוסף האילוצים. מי שאינו מכיר את תורת הסיבוכיות עשוי שלא לראות את הקושי בבעיה – "הרי המחשב מספיק מהיר כדי לבדוק את כל האפשרויות". נבחן את הטענה בהנחה שיש 40 שעות הרצאה בשבוע, 100 הרצאות שונות, ו-5 חדרים סך-הכל. בגלל שיש 100 הרצאות שונות כל משתנה יכול לקבל ערך אחד מתוך 100, ובגלל שיש

[2] לצורך הפשטות אנו מניחים שהרצאה אורכת שעה אחת בדיוק.

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רשימת אלגוריתמים

רשימת איורים

תוכן עניינים

המחקר בוצע תחת הנחיותו של פרופ' עופר שטרייכמן

רשימת פרסומים

- פרק 2 Chapter הוא עותק של המאמר [VS10a]:

Michael Veksler and Ofer Strichman. A proof-producing csp solver. In Proceedings of the Twenty-Fourth AAAI Conference on Artificial Intelligence, 2010.

תרומת המגיש: אלגוריתם לניתוח של סתירות בבעית האילוצים, הסברת האילוצים באופן עצלני, הוכחת אי-ספיקות, כללי היסק הניתנים לשימוש בהוכחת אי-ספיקות.

- דו"ח טכני [VS10b] מרחיב את [VS10a]

ומופיע ב- Appendix B:

Michael Veksler and Ofer Strichman. A proof-producing CSP solver (a proof supplement). Technical Report IE/IS-2010-02, Industrial En- gineering, Technion, Haifa, Israel, Jan 2010.

תרומת המגיש: הוכחות נכונות עבור [VS10a], הרפיית צמתים עבור פסוקיות סתירה, הסברי אילוצים מותאמים.

- פרק 3 הוא עותק של דו"ח טכני [VS14] שהוא גירסה ארוכה של מאמר בתהליך הגשה:

Michael Veksler and Ofer Strichman. Learning non-clausal constraints in csp (long version). Technical report, Technion, 2014.

תרומת המגיש: ניתוחי סתירות שאינן מבוססות על פסוקיות, כלל ההיסק Combine המשמש בניתוחי הסתירות.

- קוד המקור עבור פותר האילוצים HCSP:

`http://tx.technion.ac.il/~mveksler/HCSP/` .

תרומת המגיש: פותר האילוצים נכתב במלואו על ידי המגיש (כ-60 אלף שורות קוד). HCSP מממש את כל האלגוריתמים שפורסמו ובנוסף מממש אלגוריתם שיכול לייצר את האינטרפולנט של קרייג.

אני מודה לטכניון על התמיכה הכספית הנדיבה בהשתלמותי.

פתרון אילוצים עם מנגנון למידה מעל אילוצים כלליים

חיבור על מחקר

לשם מילוי חלקי של הדרישות לקבלת התואר
דוקטור לפילוסופיה

מיכאל ווקסלר

הוגש לסנט הטכניון — מכון טכנולוגי לישראל
יוני 2014 חיפה תמוז ה'תשע"ד

פתרון אילוצים עם מנגנון למידה מעל אילוצים כלליים

מיכאל ווקסלר